

MaCS 2010

8-th JOINT CONFERENCE ON

MATHEMATICS

AND

COMPUTER SCIENCE

Komárno, Slovakia, July 14–17, 2010

SELECTED PAPERS

EDITORS:

Horia F. Pop and Antal Bege

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Patron's Preface

The Joint Conference on Mathematics and Computer Science is an international meeting of scientists in the fields of mathematics and computer science. This conference was first initiated by Eötvös Loránd University in Budapest and Babeş-Bolyai University in Cluj-Napoca in 1994. The topics of the conference include all branches of mathematics, computer science and their applications. It is open to the interests of the potential participants from universities and research institutions.

With an increasing number of participants, several other universities joined this cooperation. The first seven Conferences took place in Ilieni, Romania in 1995 and 1997, Visegrád, Hungary in 1999, then in Băile Felix, Romania in 2001, in Debrecen, Hungary in 2004, in Pécs, Hungary in 2006 and in Cluj-Napoca, Romania in 2008. The organizer of the 8-th Joint Conference on Mathematics and Computer Science in 2010 was J. Selye University in Komárno, Slovakia. 85 participants from more than 8 countries registered to the Conference, which hopefully contributed to deepening the old friendships and initiated new ones as well as lead to an intensive exchange of scientific ideas.

I would like to thank for the opportunity to be a part of the Organising Committee of this Conference. Our university was graced with this honour.

In this preface I would like to thank the Co-Chairs of the 8th Conference on Mathematics and Computer Science, Professors:

- Petru Mocanu, from Babeş-Bolyai University in Cluj Napoca and
- Zoltán Sebestyén from Eötvös Loránd University in Budapest.

I am pleased to thank the Steering Committee of the Joint Conference on Mathematics and Computer Science :

- Antal Bege, Sapientia University, Târgu-Mureş,
- Zoltán Csörnyei, Eötvös Loránd University, Budapest,
- István Faragó, Eötvös Loránd University, Budapest,
- Zsolt Páles, University of Debrecen,
- Horia F. Pop, Babeş-Bolyai University, ClujNapoca,
- Anna Soós, Babeş-Bolyai University, ClujNapoca,
- László Tóth, University of Pécs.

I also thank the organizers of the event:

- the Faculty of Mathematics and Computer Science of the Babeş-Bolyai University in Cluj-Napoca,
- the Faculty of Informatics and the Institute of Mathematics of the Eötvös Loránd University in Budapest,
- the Department of Mathematics and Informatics of the Sapientia - Hungarian University of Transylvania,
- the Faculty of Informatics and the Institute of Mathematics of the University of Debrecen,
- the Institute of Mathematics and Informatics of the University of Pécs,
- the Bolyai Institute (Institute of Mathematics) and the Kalmár László Institute (Institute of Informatics) of the University of Szeged,
- the Department of Mathematics and the Department of Computer Science of the J. Selye University in Komárno.

This volume includes selected and refereed papers of MaCS 2010, based on invited and contributed presentations of the conference.

doc. RNDr János Tóth, PhD.

The Patron of the 8th Conference on Mathematics and Computer Science

Editors' Preface

The Joint Conference of Mathematics and Computer Science (MaCS) was started in 1994 as a series of biennial conferences by the Eötvös Lóránd University in Budapest, and the Babeş-Bolyai University in Cluj. The purpose of the conference has been to become a forum of intellectual communication and co-operation covering all branches of Mathematics, Computer Science, and their applications, open to the interests of all potential participants, from scientific academy to graduate students and industry.

The Komárno edition of MaCS is the eighth, following the successful development of the previous editions in Ilieni/Ilyefalva, Romania (1995 and 1997), Visegrád, Hungary (1999), Băile Felix/Félixfürdő, Romania (2001), Debrecen, Hungary (2004), Pécs, Hungary (2006), Cluj/Kolozsvár, Romania (2008).

Following the original effort of the universities ELTE in Budapest and BBU in Cluj, the series of MaCS conferences is now organized by an informal consortium of eight universities, namely the Babeş-Bolyai University in Cluj, the Eötvös Loránd University in Budapest, the Sapientia - Hungarian University of Transylvania, the University of Debrecen, the University of Pécs, the University of Szeged, and the János Selye University in Komárno.

This significant extension confirms without any doubt that the idea of the original creators was just. What was 16 years ago a singular effort of two partner universities is becoming a Central European conference contributing to a better understanding of Mathematics and Computer Science. This would not have been possible without the consistent efforts of all the people taking care of all the previous editions, to whom we are all indebted.

The Program Committee invited keynote lecturers from Péter P. Pálffy (ELTE), Florian Mircea Boian (BBU) and Ladislav Mišík (Ostrava) and János Tóth (JSU). Their contributions were essential to the success of our conference.

Altogether, 34 papers were selected for publication in this Conference Proceedings volume. Each submission was evaluated by one Program Committee member and two independent reviewers. We would like to thank the members

of the Program Committee for the evaluation of the submissions and the numerous referees who assisted in this work. We also thank the members of the Organizing Committee, and all the people, unheard and untold, that made the 8-th edition of the Joint Conference in Mathematics and Computer Science an enjoyable and productive reality.

July 2011

Horia F. Pop

Department of Computer Science
Babeş-Bolyai University
Cluj
Romania

Antal Bege

Department of
Mathematics and Informatics
Tg. Mureş/Marosvásárhely
Sapientia Hungarian University
of Transylvania
Cluj-Tg. Mureş-M. Ciuc
Romania

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MATHEMATICS

THE IMAGE OF THE EXPONENTIAL MAP AND SOME APPLICATIONS

DORIN ANDRICA AND RAMONA-ANDREEA ROHAN

ABSTRACT. In the first section, results on the surjectivity of some maps $M_n(K) \rightarrow M_n(K)$, where $K = \mathbb{R}$ or $K = \mathbb{C}$, are presented. The second section introduces some notions and results on Lie groups and the exponential map. The problem of finding the image of the exponential map in the general setting is stated here. As an important example we consider in the *subsection* 2.1. the exponential image of $GL(n, \mathbb{R})$, the real general linear group in dimension n . Section 2.2. is devoted to the Lie groups $SO_k(n)$. A new proof to show that it has a smooth structure and to find its dimension it is given (*Theorem 2.2.1*). The exponential image of the group $SO(2, 1)$ is determined by using two methods in Section 2.3.

1. RESULTS ON THE SURJECTIVITY OF SOME MAPS $M_n(K) \rightarrow M_n(K)$

Let $M_n(K)$ be the algebra of the square matrices with entrees in the commutative field K ($K = \mathbb{R}$ or $K = \mathbb{C}$). Consider the convergent power series with coefficients in K , $\sum_{m \geq 0} a_m z^m$, $z \in K$. Let

$$f(z) = \sum_{m=0}^{\infty} a_m z^m, \quad z \in K.$$

For $n \geq 2$ and for $X \in M_n(K)$ we know that the series $\sum_{m=0}^{\infty} a_m X^m$ is also convergent and we denote its sum by $\tilde{f}(X)$.

The following two problems are important.

Problem 1. *What are the conditions on function f such that the operator $\tilde{f}$ is surjective?*

Problem 2. *If the operator $\tilde{f}$ is not surjective, then describe its range.*

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These problems are strongly connected to the solvability of some matrix equations of the form $f(X) = A$. We refer to the monograph [1, Chapter 1] for historical references and the evolution of the problems.

Concerning Problem 1, we mention that in the case when $K = \mathbb{R}$ and f is a polynomial of even degree, the operator $\tilde{f}$ is not surjective. If the polynomial f is of odd degree Problem 2 is difficult and it is still open (we also refer to [1, Chapter 1] for details in this direction).

Even that in the case $K = \mathbb{R}$ the considered problems are difficult, we have good enough results for $K = \mathbb{C}$.

We shall review few such results connected to our problems.

A complete characterization of the surjectivity of $\tilde{f}$ is the following and it had been proved for the case of polynomials by S. Rădulescu, D. Andrica [24] and in the general situation by D. Andrica and L. Mare [3].

Theorem 1.1. *Let $f : \mathbb{C} \rightarrow \mathbb{C}$ a holomorphic function and $n \in \mathbb{N}$, $n \geq 2$. The operator $\tilde{f} : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ is surjective if and only if $\forall \alpha \in \mathbb{C}$ the equation $f(z) = \alpha$ has at least one simple root.*

For instance, the operator $\tilde{f} : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$, $\tilde{f}(X) = X^m$, is not surjective, where $m \geq 2$. Indeed, the equation $x^m = 0$ has 0 as multiple zero of multiplicity m , hence the property directly follows from the previous Theorem 1.1.

An interesting situation is given by the exponential map

$$\exp(z) = \sum_{m=0}^{\infty} \frac{1}{m!} z^m, \quad z \in K,$$

and we easily derive the following result.

Corollary 1.1 *The operator $\exp = \tilde{\exp} : M_n(\mathbb{C}) \rightarrow GL(n, \mathbb{C})$ is surjective.*

Since the field of real numbers is not algebraic closed Problem 1 is much more difficult in the case $K = \mathbb{R}$ than $K = \mathbb{C}$. We have the following result proved by L. Mare [8]:

Theorem 1.2. *Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ be a powers series with real coefficients and the convergency radius ∞ , $n \in \mathbb{N}$, $n \geq 2$.*

I. If $n \in \{2, 3\}$, then $\tilde{f} : M_n(\mathbb{R}) \rightarrow M_n(\mathbb{R})$ is surjective if and only if:

(ii) $\forall \mu \in \mathbb{R}$ the equation $f(x) = \mu$ has at least a simple real root.

(ii) $f(\mathbb{C}) = \mathbb{C}$.

II. If $n \geq 4$, then $\tilde{f} : M_n(\mathbb{R}) \rightarrow M_n(\mathbb{R})$ is surjective if and only if:

(i)' $\forall \mu \in \mathbb{R}$ the equation $f(x) = \mu$ has at least a simple real root.

(ii)' $\forall \mu \in \mathbb{C}$ the equation $f(z) = \mu$ has at least a simple root in $\mathbb{C}$.

The study of the exponential map is also much more complicated in this case. A first remark is that we have the inclusion:

$$\exp(M_n(\mathbb{R})) \subseteq GL^+(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) : \det A > 0\}.$$

Some results related to the image of the exponential map in this case will be discussed in the next section.

A partial answer to Problem 2 in case $K = \mathbb{C}$ is given by L. Mare [8] in the following result:

Theorem 1.3. *Let $f : \mathbb{C} \rightarrow \mathbb{C}$ a holomorphic function and $n \in \mathbb{N}$, $n \geq 2$ and $A \in M_n(\mathbb{C})$, $\sigma(A) = \{\alpha_1, \dots, \alpha_s\}$ the spectrum of matrix A , If for every $j \in \{1, 2, \dots, s\}$, the equation $f(z) = \alpha_j$ has at least a simple root, then $A \in \text{Im} \left(\tilde{f} \right)$.*

2. LIE GROUPS OF MATRICES AND THE EXPONENTIAL MAP

Let G be a Lie group with its Lie algebra $L(G)$. It is well known that the exponential map $\exp : L(G) \rightarrow G$ is defined by $\exp(X) = \gamma_X(1)$, where $X \in L(G)$ and γ_X is the one-parameter subgroup of G induced by X . Recall the following properties of the exponential map:

- 1) For any $t \in \mathbb{R}$ and for any $X \in L(G)$ we have $\gamma_X(t) = \exp(tX)$;
- 2) For any $s, t \in \mathbb{R}$ and for any $X \in L(G)$, we have $\exp(sX)\exp(tX) = \exp(s+t)X$;
- 3) For any $t \in \mathbb{R}$ and for any $X \in L(G)$, we have $\exp(-tX) = (\exp tX)^{-1}$;
- 4) $\exp : L(G) \rightarrow G$ is a smooth mapping, it is a local diffeomorphism at $0 \in L(G)$ and $\exp(0) = e$, where e is the unity element of the group G ;
- 5) The image $\exp(L(G))$ of the exponential map generates the connected component G_e of the unity $e \in G$;
- 6) If $f : G_1 \rightarrow G_2$ is a morphism of Lie groups and $f_* : L(G_1) \rightarrow L(G_2)$ is the induced morphism of Lie algebras by f , then $f \circ \exp_1 = \exp_2 \circ f_*$.

As we can note from the previous property 5, the following problem is of special importance:

Problem. *If G is a Lie group with Lie algebra g and exponential map $\exp : L(G) \rightarrow G$, find the set $E(G) = \exp(g)$.*

J. Dixmier has proposed first time the problem to find $E(G)$ for the resoluble connected Lie groups. Only in few special situations we have $G = E(G)$,

and the groups satisfying this property are called *exponential Lie groups*. The compact connected Lie groups are exponential (see [6, Theorem 2.2, pp. 165] or the monograph [2]). This property is satisfied by many Lie groups of rank 1 having simple connected free center ([11] and [12]). The problem is very complicated for semisimple Lie groups of rank ≥ 2 . For details we recommend the monograph [25].

Recall that the Lie group $\text{Lie } G$ is semisimple if it has no nontrivial connected normal Abelian subgroups.

Concerning the considered problem we mention that H.L. Lai [7] has proved that if G is real, semisimple and with finite center, then there is a positive integer $p = p(G)$ such that $g^p \in E(G)$ for any $g \in L(G)$, where $L(G)$ is the Lie algebra of G . He obtained by using this result the surjectivity of the exponential map for some classes of real Lie groups.

2.1. The exponential image on $GL(n, \mathbb{R})$. In this section we shall discuss the problem of finding the exponential image for real general linear group $G = GL(n, \mathbb{R})$. It is known that $GL(n, \mathbb{R})$ has two connected components, that is

$$GL^-(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) : \det A < 0\}$$

and

$$GL^+(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) : \det A > 0\}.$$

The Lie algebra of $GL(n, \mathbb{R})$ is $M_n(\mathbb{R})$, the algebra of square matrices of order n with real entries and the exponential map is defined by $\exp : M_n(\mathbb{R}) \rightarrow GL(n, \mathbb{R})$, where

$$\exp(A) = \sum_{k=0}^{\infty} \frac{1}{k!} A^k.$$

For $n = 1$, we identify $M_1(\mathbb{R}) = \mathbb{R}$ and $GL^+(1, \mathbb{R}) = (0, +\infty)$, hence $\exp : \mathbb{R} \rightarrow (0, +\infty)$ is surjective. Therefore the problem is interesting if $n \geq 2$.

In this case we will present the results contained in the paper [3] (see also the monograph [2]).

Theorem 2.1.1. ([3]) *Consider the matrix $A \in GL^+(n, \mathbb{R})$. Then $A \in \exp(M_n(\mathbb{R}))$ if and only if the Jordan blocks corresponding to the negative eigenvalues in its Jordan decomposition appear with even multiplicities.*

This result was first time mention in the papers [20] and [21] of T.Nono (see also [14]).

We present now a different characterization of the exponential image of the Lie group $GL(n, \mathbb{R})$, $n \geq 2$.

Theorem 2.1.2. *Let $A \in GL(n, \mathbb{R})$. Then $A \in \exp(M_n(\mathbb{R}))$ if and only if $A = B^2$ for some matrix $B \in GL(n, \mathbb{R})$.*

Remarks. 1) The topology of the set $E(G)$ is very interesting. Generally, $E(G)$ is not open and it is not closed in G . We shall illustrate this for the Lie group $SL(2, \mathbb{R})$ by using the idea in paper [9]. In order to show that the set $E(SL(2, \mathbb{R}))$ is not open, consider the matrix

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \exp \begin{pmatrix} 0 & \pi \\ -\pi & 0 \end{pmatrix}.$$

We have that, for any $\varepsilon > 0$, the matrix $\begin{pmatrix} -1 & \varepsilon \\ 0 & -1 \end{pmatrix}$ does not belong to $E(SL(2, \mathbb{R}))$ because the number of blocks corresponding to the eigenvalue -1 is equal to 1 (Theorem 2.2.1.). Let us note that $E(SL(2, \mathbb{R}))$ is not closed. In this respect let consider the matrix

$$\begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix} \in SL(2, \mathbb{R}) \setminus E(SL(2, \mathbb{R}))$$

and let observe that for any $\varepsilon > 0$ we have

$$\begin{pmatrix} -1 & -1 \\ \varepsilon & -1 \end{pmatrix} \in E(SL(2, \mathbb{R})).$$

Hence, if $E(SL(2, \mathbb{R}))$ is closed, then we get

$$\lim_{\varepsilon \searrow 0} \begin{pmatrix} -1 & -1 \\ \varepsilon & -1 \end{pmatrix} \in E(SL(2, \mathbb{R})),$$

which is not possible.

2) The interior and the boundary of $E(GL(n, \mathbb{R}))$ are determined in the paper [20] (see also [14]), and the results are:

- i) $A \in \text{int}E(GL(n, \mathbb{R}))$ if and only if A has no negative eigenvalues;
- ii) A is a boundary point of $E(GL(n, \mathbb{R}))$ if and only if $A \notin \text{int}(E(GL(n, \mathbb{R})))$

and the corresponding blocks to the negative eigenvalues in the Jordan decomposition appear with even multiplicities.

2.2. The Lie groups $SO_K(n)$. Let $K \in M_n(\mathbb{R})$ be a symmetric nonsingular matrix. Define

$$SO_K(n) = \{A \in M_n(\mathbb{R}) : {}^tAKA = K\}.$$

Note that $(SO_K(n), \cdot)$ is a subgroup of $GL(n, \mathbb{R})$. Indeed, if $A, B \in SO_K(n)$, then we can write

$${}^t(AB)K(AB) = {}^tB {}^tAKAB = {}^tB({}^tAKA)B = {}^tBKB = K,$$

hence $AB \in SO_K(n)$. Moreover, if $A \in SO_K(n)$, then

$${}^t(A^{-1})KA^{-1} = ({}^tA)^{-1}KA^{-1} = KAA^{-1} = K,$$

that is $A^{-1} \in SO_K(n)$.

Remark that for $K \in M_n(\mathbb{R})$ we can consider the bilinear form $f_K : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$, defined by $f_K(x, y) = {}^txKy$. It follows that for $A \in SO_K(n)$ we have

$$f_K(Ax, Ay) = {}^t(Ax)K(Ay) = {}^tx({}^tAKA)y = {}^txKy = f_K(x, y),$$

hence the set $SO_K(n)$ consists in all nonsingular matrices A satisfying the relation $f_K(Ax, Ay) = f_K(x, y)$.

Recall the fact that the matrices $K, K' \in M_n(\mathbb{R})$ are equivalent, that is $K \sim K'$, if $K' = {}^tUKU$, for some $U \in GL(n, \mathbb{R})$. In this case we have $SO_K(n) \simeq SO_{K'}(n)$. Indeed, the general linear group $GL(n, \mathbb{R})$ acts on $M(\mathbb{R})$ by $UA = {}^tUAU$ and $SO_K(n)$ is the isotopy group corresponding to matrix K . From $K \sim K'$ it follows that the matrices K and K' belong on the same orbit with respect to this action, hence necessarily $SO_K(n) \simeq SO_{K'}(n)$. In fact these subgroups are conjugate in $GL(n, \mathbb{R})$. We refer to [5] and [10] for details involving this notions and for connection to other matrices groups.

Using the well known Sylvester Theorem, it follows that the matrix K is equivalent to a diagonal matrix of the form

$$\text{diag}(\underbrace{1, \dots, 1}_r, \underbrace{-1, \dots, -1}_{n-r}),$$

where r represents the *signature* of K . In this case the group $SO_K(n)$ is also denoted by $SO(r, n-r)$.

It is clear that $SO(r, n-r) = SO(n-r, r)$, therefore it is important to consider only the situation $r \leq \lfloor \frac{n}{2} \rfloor$. The integer r is an invariant of the group $SO(r, n-r)$.

Recall that $SO(1, 3)$ is called the *Lorentz group* and it is very important in the Theory of Relativity.

Another important case is given by the situation when K is equivalent to the unity matrix I_n . In this case, $SO_K(n)$ is isomorphic to $SO(n)$, the *real special orthogonal group*.

When $n = 2m$ and

$$K = \left(\begin{array}{c|c} 0 & -I_m \\ \hline - & - \\ I_m & 0 \end{array} \right)$$

then $SO_K(n)$ becomes the *symplectic group* $Sp_{2m}(\mathbb{R})$, which is important in Classical Mechanics and Geometric Optics.

In what follows we will show that $SO_K(n)$ has a structure of Lie group. In this respect we consider the continuous map $h : GL(n, \mathbb{R}) \rightarrow GL(n, \mathbb{R})$, defined by $h(A) = {}^tAKA$. We have $SO_K(n) = h^{-1}(K)$, hence $SO_K(n)$ is a closed subset of $GL(n, \mathbb{R})$. It follows that $SO_K(n)$ is smooth submanifold of $GL(n, \mathbb{R})$ (see [6, Theorem 3.11, pp. 28]).

In order to find the dimension of the submanifold $SO_K(n)$ we shall use the method given in the paper [4] (see also [1, pp. 45-46]). First of all observe that ${}^th(A) = {}^t({}^tAKA) = {}^tAKA^K = h(A)$, hence

$$h : GL(n, \mathbb{R}) \rightarrow S_n(\mathbb{R}),$$

where $S_n(\mathbb{R})$ the set of all symmetric matrices of order n with real entrees. In addition, we have

$$\begin{aligned} (dh)_X(Y) &= \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} (h(X + \lambda Y) - h(X)) \\ &= \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} ({}^t(X + \lambda Y)K(X + \lambda Y) - {}^tXKX) \\ &= \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} (({}^tX + \lambda {}^tY)K(X + \lambda Y) - {}^tXKX) = {}^tXKY + {}^tYKX. \end{aligned}$$

We shall show that for any matrix $X \in SO_K(n)$, the differential $(dh)_X : M_n(\mathbb{R}) \rightarrow S_n(\mathbb{R})$ is surjective, that is the rank of the map h at X is $\frac{n(n+1)}{2}$. Let $Z \in S_n(\mathbb{R})$. The relation $(dh)_X(Y) = Z$ holds, for example for $Y = \frac{1}{2}K{}^t(X^{-1})K^{-2}Z$.

Indeed, in this case we have

$$\begin{aligned} (dh)_X(Y) &= \frac{1}{2} {}^tXKK{}^t(X^{-1})K^{-2}Z + \frac{1}{2} {}^tZ{}^t(K^{-2}){}^t(X^{-1})KX \\ &= \frac{1}{2} KX^{-1}K{}^t(X^{-1})K^{-2}Z + \frac{1}{2} ZK^{-2}X^{-1}KX \\ &= \frac{1}{2}Z + \frac{1}{2}ZK^{-2}X^{-1}K(X^{-1}){}^tXKX = \frac{1}{2}Z + \frac{1}{2}Z = Z. \end{aligned}$$

It follows that $Z \in S_n(\mathbb{R})$ is a regular value of h , hence h is a submersion at X . Applying the pre-image Theorem we get that $SO_K(n)$ is a submanifold of dimension $n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$ of $GL(n, \mathbb{R})$.

To find the Lie algebra of $SO_K(n)$, let us consider the smooth curve $c : \mathbb{R} \rightarrow SO_K(n)$ cu $c(0) = I_n$. From the relation $K = {}^t c(s) K c(s)$, taking the derivative, we get

$$O_n = {}^t(\dot{c}(0))Kc(0) + {}^t c(0)K\dot{c}(0) = {}^t LK + KL,$$

where $L = \dot{c}(0)$. The matrix K is symmetric, hence we can write $O_n = {}^t(KL) + KL$. It follows in this way that the Lie algebra of $SO_K(n)$ is

$$so_K(n) = \{L \in M_n(\mathbb{R}) : KL \text{ is skew-symmetric}\} = K \cdot AS_n(\mathbb{R}),$$

where $AS_n(\mathbb{R})$ represents the set of all skew-symmetric matrices of order n with real entrees. In this way we have obtained the following result:

Theorem 2.2.1. *For any non-singular symmetric matrix $K \in M_n(\mathbb{R})$, $SO_K(n)$ is a Lie group of dimension $\frac{n(n-1)}{2}$ having the Lie algebra $so_K(n) = K \cdot AS_n(\mathbb{R})$.*

The Lie groups $SO_K(n)$ are of the special importance in Geometric Mechanics (see the monographs [2] and [13]).

For $A \in SO_K(n)$, from the relation ${}^t A K A = K$ by passing to the determinants we get $\det^2 A = 1$, hence $\det A = \pm 1$. It follows that the Lie group $SO_K(n)$ has two connected components, namely

$$SO_K^+(n) = \{A \in SO_K(n) : \det A = 1\}$$

and

$$SO_K^-(n) = \{A \in SO_K(n) : \det A = -1\}.$$

2.3. The exponential image of $SO(2, 1)$. In the first part of this subsection we shall compute in an explicit way the exponential image of the Lie group $SO(2, 1)$. According to the main result in the previous subsection, for any $A \in so(2, 1)$, we have

$$A = \begin{pmatrix} 0 & b & c \\ b & 0 & a \\ c & -a & 0 \end{pmatrix}.$$

The characteristic polynomial of A is given by :

$$P_A(x) = -x^3 + (b^2 + c^2 - a^2)x.$$

We have to consider three cases:

Case 1. $u = b^2 + c^2 - a^2 > 0$

Then, the eigenvalues of A are :

$$x_1 = 0, \quad x_{2,3} = \pm\sqrt{u}$$

But

$$e^x = P_A(x)f(x) + a_0 + a_1x + a_2x^2 \quad (1)$$

hence

$$e^A = a_0I_3 + a_1A + a_2A^2.$$

Considering in (1) $x_1 = 0$, $x_2 = \sqrt{u}$, $x_3 = -\sqrt{u}$ we find the following expression for $\exp(A)$:

$$\exp(A) = I_3 + \frac{\text{sh}(\sqrt{u})}{\sqrt{u}}A + \frac{\text{ch}(\sqrt{u}) - 1}{u}A^2.$$

Case 2. $u = b^2 + c^2 - a^2 = 0$

In the similar way we get:

$$\exp(A) = I_3 + A + \frac{1}{2}A^2$$

Case 3. $u = b^2 + c^2 - a^2 < 0$

In the similar way it follows:

$$\exp(A) = I_3 + \frac{\sin(\sqrt{-u})}{\sqrt{-u}}A + \frac{\cos(\sqrt{-u}) - 1}{u}A^2.$$

This is a direct way to prove the result in the paper [22].

Theorem 2.3.1. For any $A \in so(2, 1)$

$$A = \begin{pmatrix} 0 & b & c \\ b & 0 & a \\ c & -a & 0 \end{pmatrix}$$

we have:

$$\exp(A) = \begin{cases} I_3 + \frac{\text{sh}(\sqrt{u})}{\sqrt{u}}A + \frac{\text{ch}(\sqrt{u})-1}{u}A^2 & \text{if } u > 0 \\ I_3 + A + \frac{1}{2}A^2 & \text{if } u = 0 \\ I_3 + \frac{\sin(\sqrt{-u})}{\sqrt{-u}}A + \frac{\cos(\sqrt{-u})-1}{u}A^2 & \text{if } u < 0 \end{cases},$$

where $u = b^2 + c^2 - a^2$.

The same result can be obtained if we use the E.J.Putzer method [23].

We shall illustrate this method only in the case $u = b^2 + c^2 - a^2 > 0$. In this respect we shall consider few steps:

Step 1. Consider a fundamental basis for the scalar differential equation :

$$\ddot{x} - u\dot{x} = 0.$$

Since

$$u = b^2 + c^2 - a^2 > 0$$

we have:

$$\{\varphi_1(t), \varphi_2(t), \varphi_3(t)\} = \{1, e^{t\sqrt{u}}, e^{-t\sqrt{u}}\}.$$

Step 2. Consider the matrix:

$$M = \begin{pmatrix} \varphi_1(0) & \varphi_2(0) & \varphi_3(0) \\ \dots & \dots & \dots \\ \varphi_1(0) & \varphi_2(0) & \varphi_3(0) \\ \dots & \dots & \dots \\ \varphi_1(0) & \varphi_2(0) & \varphi_3(0) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -\sqrt{u} & \sqrt{u} \\ 0 & u & u \end{pmatrix}.$$

Step 3. Compute the matrix:

$$\begin{aligned} (f_1(t), f_2(t), f_3(t)) &= (\varphi_1(t), \varphi_2(t), \varphi_3(t))M^{-1} \\ &= (1, e^{t\sqrt{u}}, e^{-t\sqrt{u}}) \begin{pmatrix} 1 & 0 & -\frac{1}{u} \\ 0 & -\frac{1}{2\sqrt{u}} & \frac{1}{2u} \\ 0 & \frac{1}{2\sqrt{u}} & \frac{1}{2u} \end{pmatrix} \\ &= \left(1, \frac{\text{sh}(t\sqrt{u})}{\sqrt{u}}, \frac{\text{ch}(t\sqrt{u}) - 1}{u}\right). \end{aligned}$$

Step 4. We have:

$$\begin{aligned} \exp(At) &= f_1(t)I_3 + f_2(t)A + f_3(t)A^2 \\ &= I_3 + \frac{\text{sh}(t\sqrt{u})}{\sqrt{u}}A + \frac{\text{ch}(t\sqrt{u}) - 1}{u}A^2 \end{aligned}$$

Step 5. Considering $t = 1$, we get:

$$\exp(A) = I_3 + \frac{\text{sh}(\sqrt{u})}{\sqrt{u}}A + \frac{\text{ch}(\sqrt{u}) - 1}{u}A^2.$$

The cases $u = 0$ and $u < 0$ are similar.

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FACULTY OF MATHEMATICS AND COMPUTER SCIENCE, BABES-BOLYAI UNIVERSITY,
CLUJ-NAPOCA, ROMANIA

E-mail address: dandrica@math.ubbcluj.ro, ramona.rohan@ubbcluj.ro

HYPERPERFECT NUMBERS AND GENERALIZATIONS

ZSOLT BARTHA AND ANTAL BEGE

ABSTRACT. Let $\sigma(n)$ denote the sum of positive divisors of the natural number n . A natural number is perfect if $\sigma(n) = 2n$. This concept was already generalized in the form of hyperperfect numbers

$$\sigma(n) = \frac{k+1}{k}n + \frac{k-1}{k},$$

and super-hyperperfect numbers

$$\sigma(\sigma(n)) = \frac{k+1}{k}n + \frac{k-1}{k}.$$

In this paper we present some new results, numerical results and establish some conjectures.

1. Introduction

For the natural number n we denote the sum of positive divisors by

$$\sigma(n) = \sum_{d|n} d, \quad s(n) = \sum_{d|n, d \neq n} d.$$

Definition 1. A *perfect number* is a number that is half the sum of all its positive divisors

$$\sigma(n) = 2n, \quad \text{or} \quad s(n) = n.$$

The first perfect number is 6, and the other perfect numbers are

$$28, 496, 8128, 33550336, \dots$$

Euclid discovered that the first four even perfect numbers are generated by the formula

$$n = 2^{p-1}(2^p - 1)$$

where p and $2^p - 1$ are prime numbers. Euler proved that this formula will yield all the even perfect numbers.

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Definition 2. A *Mersenne prime* is a prime number of the forme

$$M_n = 2^{p_n} - 1$$

where p_n must also be a prime number.

There is a one-to-one relationship between even perfect numbers and Mersenne primes. The 47th Mersenne prime was discovered in 2009 [15, 14]:

$$2^{43112609} - 1$$

and therefore we have the following even perfect number

$$2^{43112608}(2^{43112609} - 1).$$

It is an unsolved problem whether there are any odd perfect numbers?

We know that any odd perfect number must satisfy the following conditions:

- Any odd perfect number must be of the form $12m + 1$ or $36m + 9$.
- If n is an odd perfect number, it has the following form

$$n = p^\alpha m^2, \quad p \equiv \alpha \equiv 1(\text{mod } 4), \quad p \in \mathbf{P}.$$

- If n is an odd perfect number, then

$$n \geq 10^{300}.$$

D. Suryanarayana introduced the notion of superperfect number in 1969 [13].

Definition 3. A natural number n is called *superperfect number* if

$$\sigma^2(n) = \sigma(\sigma(n)) = 2n.$$

Some properties

- If n is a superperfect number, then

$$n = 2^{p-1}, \quad p, 2^{p-1} - 1 \in \mathbf{P} \text{ [5].}$$

- If odd superperfect numbers exist, they are square numbers.
- If n is an odd superperfect number, then

$$n > 7 \cdot 10^{24} \text{ [13].}$$

2. Hyperperfect numbers

Minoli and Bear introduced the notion of hyperperfect number in 1975 [8]. Hyperperfect numbers are the generalization of perfect numbers.

Definition 4. A natural number n is called *k-hyperperfect number* if

$$n = 1 + k \sum_{d|n, 1 < d < n} d$$

where the summation is taken over all proper divisors, so that

$$n = 1 + k[\sigma(n) - n - 1]$$

and rearranging gives

$$\sigma(n) = \frac{k+1}{k}n + \frac{k-1}{k}.$$

For $k = 1$ the 1-hyperperfect numbers are the classical perfect numbers.

$$\sigma(n) = 2n.$$

For other results see [1, 3, 4, 6, 8].

For $k = 2$ and $k = 4$ we obtain

$$\sigma(n) = \frac{3}{2}n + \frac{1}{2}, \quad \sigma(n) = \frac{5}{4}n + \frac{3}{4}.$$

Lemma 5. *We have the following results:*

- If $n = 3^{l-1}(3^l - 2)$ where $3^l - 2$ is prime, then n is a 2-hyperperfect number.
- If $n = 5^{l-1}(5^l - 4)$ where $5^l - 4$ is prime, then n is a 4-hyperperfect number.

Proof. Since

$$(3^{l-1}, 3^l - 2) = 1, \quad (5^{l-1}, 5^l - 4) = 1$$

and the divisor function σ is multiplicativ we have

$$\begin{aligned} \sigma(n) &= \sigma(3^{l-1}(3^l - 2)) = \sigma(3^{l-1}) \cdot \sigma(3^l - 2) = \frac{3^{(l-1)+1} - 1}{3 - 1} \cdot (3^l - 2 + 1) = \\ &= \frac{3}{2}3^{l-1}(3^l - 2) + \frac{1}{2} = \frac{3}{2}n + \frac{1}{2}, \end{aligned}$$

and

$$\begin{aligned} \sigma(n) &= \sigma(5^{l-1}(5^l - 4)) = \sigma(5^{l-1}) \cdot \sigma(5^l - 4) = \frac{5^{(l-1)+1} - 1}{5 - 1} \cdot (5^l - 4 + 1) = \\ &= \frac{5}{4}5^{l-1}(5^l - 4) + \frac{3}{4} = \frac{5}{4}n + \frac{3}{4}. \end{aligned}$$

Conjecture 6. *All 2-hyperperfect numbers are of the form $n = 3^{l-1}(3^l - 2)$, where $3^l - 2$ is prime.*

We put the following problem:

For which values of k exist k -hyperperfect numbers?

The partial answer:

For $k = q - 1$, q prime, we have

$$\sigma(n) = \frac{q}{q-1}n + \frac{q-2}{q-1},$$

and John C.M. Nash [6] showed that if $n = q^l(q^{l+1} - (q-1))$, where $q^{l+1} - (q-1)$ is prime, then n is a $(q-1)$ -hyperperfect number.

3. Super-hyperperfect numbers

We introduced other new forms of generalization of hyperperfect numbers.

Definition 7. We say that a natural number n is k -**super-hyperperfect** if

$$\sigma(\sigma(n)) = \frac{k+1}{k}n + \frac{k-1}{k}.$$

For $k=1$ the 1-super-hyperperfect numbers are the classical superperfect numbers

$$\sigma(\sigma(n)) = 2n.$$

For $k = 2$ and $k = 4$ we obtain

$$\sigma(\sigma(n)) = \frac{3}{2}n + \frac{1}{2}, \quad \sigma(\sigma(n)) = \frac{5}{4}n + \frac{3}{4}.$$

Lemma 8. We have

- If $n = 3^{p-1}$ where p and $\frac{3^p-1}{2}$ are primes, then n is a 2-super-hyperperfect number.
- If $n = 5^{p-1}$ where p and $\frac{5^p-1}{4}$ are primes, then n is a 4-super-hyperperfect number.

Proof. We have

$$\begin{aligned} \sigma(\sigma(n)) &= \sigma(\sigma(3^{p-1})) = \sigma\left(\frac{3^p-1}{2}\right) = \frac{3^p+1}{2} = \\ &= \frac{3}{2} \cdot 3^{p-1} + \frac{1}{2} = \frac{3}{2}n + \frac{1}{2}, \end{aligned}$$

and

$$\begin{aligned} \sigma(\sigma(n)) &= \sigma(\sigma(5^{p-1})) = \sigma\left(\frac{5^p-1}{4}\right) = \frac{5^p+3}{4} = \\ &= \frac{5}{4} \cdot 5^{p-1} + \frac{3}{4} = \frac{5}{4}n + \frac{3}{4}. \end{aligned}$$

For $k = q - 1$, q prime, we obtain

$$\sigma(\sigma(n)) = \frac{q}{q-1}n + \frac{q-2}{q-1}$$

and if $n = q^{p-1}$ where p and $\frac{q^p-1}{p-1}$ are primes, then n is a $(q-1)$ -super-hyperperfect number.

For $k = 3$ and $k = 5$ we haven't found any numerical solution.

We put the following problems:

For which values of k exist k -super-hyperperfect numbers?

Do k -super-hyperperfect numbers with three prime divisors exist?

Theorem 9. *In general if $k = q - 1$, with q prime and n divisible by q , then all $(q - 1)$ -super-hyperperfect numbers are of the form $n = q^{p-1}$ where p and $\frac{q^p-1}{q-1}$ are primes.*

Proof. We note that, since

$$\sigma(n) = \sum_{d|n} d = n \sum_{d|n} \frac{1}{d},$$

we have

$$\sigma(\sigma(n)) = n \sum_{d|n} \frac{1}{d} \sum_{e|\sigma(n)} \frac{1}{e}.$$

Suppose n is $(q-1)$ -super-hyperperfect number with q prime and $q^l|n$ but $q^{l+1} \nmid n$.

Then

$$\sigma(\sigma(n)) = \frac{q}{q-1}n + \frac{q-2}{q-1}$$

and we obtain

$$\begin{aligned}
q &= (q-1) \cdot \frac{\sigma(\sigma(n))}{n} - \frac{q-2}{n} = (q-1) \sum_{d|n} \frac{1}{d} \sum_{e|\sigma(n)} \frac{1}{e} - \frac{q-2}{n} \geq \\
&\geq (q-1) \cdot \left(1 + \frac{1}{q} + \frac{1}{q^2} + \dots + \frac{1}{q^l}\right) \cdot \left(1 + \frac{q-1}{q^{l+1}-1}\right) - \frac{q-2}{q^l} = \\
&= (q-1) \cdot \frac{1 - \frac{1}{q^{l+1}}}{1 - \frac{1}{q}} \cdot \frac{q^{l+1} + q - 2}{q^{l+1} - 1} - \frac{q-2}{q^l} = \\
&= (q-1) \cdot \frac{q}{q-1} \cdot \frac{q^{l+1} - 1}{q^{l+1}} \cdot \frac{q^{l+1} + q - 2}{q^{l+1} - 1} - \frac{q-2}{q^l} = \\
&= \frac{q^{l+1} + q - 2}{q^l} - \frac{q-2}{q^l} = \frac{q^{l+1}}{q^l} = q.
\end{aligned}$$

We must have equality throughout. Thus $n = q^{p-1}$ where p and $\frac{q^p - 1}{q - 1}$ are primes.

Collorary 10. • *All 2-super-hyperperfect numbers divisible by 3 are of the form $n = 3^{p-1}$ where p and $\frac{3^p - 1}{2}$ are primes.*
• *All 4-super-hyperperfect numbers divisible by 5 are of the form $n = 5^{p-1}$ where p and $\frac{5^p - 1}{4}$ are primes.*

4. Numerical results

For finding the numerical results we have used a Maple program.
Some known k -hyperperfect numbers for different k values

- 2-hyperperfect numbers: 21, 2133, 19521, 176661, 129127041, ...

$$21 = 3 \cdot 7 = 3 \cdot (3^2 - 1)$$

$$2133 = 3^3 \cdot 79 = 3^3 \cdot (3^4 - 2)$$

$$19521 = 3^4 \cdot 241 = 3^4 \cdot (3^5 - 2)$$

- 3-hyperperfect numbers: 325, ...

$$325 = 5^2 \cdot 13$$

- 4-hyperperfect numbers: 1950625, 1220640625, ...
- 6-hyperperfect numbers: 301, 16513, 60110701, ...

$$301 = 7 \cdot 43 = 7 \cdot (7^2 - 6)$$

- 10-hyperperfect numbers: 159841, ...

- 12-hyperperfect numbers: 697, 2041, 1570153, 62722153, ...

We have searched 2-super-hyperperfect numbers for primes p , when $\frac{3^p - 1}{2}$ is also prime.

We obtained the following results

$$\begin{aligned} p = 3, \frac{3^3 - 1}{2} \in \mathbf{P} : \quad n_1 &= 3^2 \\ p = 7, \frac{3^7 - 1}{2} \in \mathbf{P} : \quad n_2 &= 3^6 \\ p = 13, \frac{3^{13} - 1}{2} \in \mathbf{P} : \quad n_3 &= 3^{12} \end{aligned}$$

The last result is

$$p = 9551, \frac{3^{9551} - 1}{2} \in \mathbf{P} : \quad n_9 = 3^{9550}.$$

5. ACKNOWLEDGEMENT

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SAPIENTIA UNIVERSITY, DEPARTMENT OF MATHEMATICS AND INFORMATICS, TÂRGU
MUREŞ, ROMANIA

E-mail address: `zsoltb@ms.sapientia.ro`, `abege@ms.sapientia.ro`

LIST COLORING OF LATIN AND SUDOKU GRAPHS

ANTAL IVÁNYI AND ZSOLT NÉMETH

ABSTRACT. In 2001 Galvin [37] proved a generalization of Hall theorem [41]. We improve his result in two directions, giving a weaker precondition and also a stronger consequence. We show some basic properties of the Latin and Sudoku graphs considering the solution of the Latin and Sudoku puzzles as a list coloring problem of graphs [9, 11, 16, 27, 32, 48, 49, 51, 53, 74].

1. INTRODUCTION

List coloring of graphs was proposed by Vizing in 1976 [77]. Substantial progress was made by Erdős, Rubin and Taylor in 1979 [25].

Today graph coloring is a popular research topic of combinatorics and computer science. Concrete applications appear in connection with VLSI design [52], networks [31, 56, 63], resource allocation [38], and determination of partially given Latin [28] and Sudoku [14, 15, 33, 54, 55] squares.

Let $m \geq 1$ and $n \geq 2$ be integers and $G = (V, E)$ be a finite simple graph with vertex set $V = \{v_1, v_2, \dots, v_n\}$ and edge set $E = \{e_1, e_2, \dots, e_m\}$. Assume further that an infinite set $\mathcal{C} = \{c_1, c_2, \dots\}$ of so called *colors*, and a collection $\mathcal{L} = \{L(v_1), L(v_2), \dots, L(v_n)\}$ of lists of colors are given, where $L(v_i) \subseteq \mathcal{C}$ for $1 \leq i \leq n$. Here each list $L(v_i)$ is considered as the set of allowed colors for vertex v_i . $\mathcal{L}$ is called *list assignment*.

For the simplicity and without the loss the generality we suppose that $\mathcal{C}$ is the set of positive integers, and so each list is a finite set of positive integers. If the sizes of the lists have a join upper bound, that is if $|L(v_i)| \leq k$, then the collection $\mathcal{L}$ is called a *k-assignment* on G . Since the components of G are colorable independently, for the simplicity we suppose that G is connected.

A sequence of colors (positive integers) $S = (c_{i_1}, c_{i_2}, \dots, c_{i_n}) = (i_1, i_2, \dots, i_n)$ is called a *proper vertex coloring* of G with a given collection of lists $\mathcal{L}$, if for all $1 \leq j < k \leq n$

- (1) $i_j \in L(v_j)$;

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(2) if $v_j v_k \in E$, then $i_j \neq i_k$.

Graph G with a given $\mathcal{L}$ is $\mathcal{L}$ -colorable (list colorable), if there exists a proper vertex coloring function $\varphi : V \rightarrow \mathcal{C}$ satisfying for all $1 \leq j < k \leq n$

(1) $\varphi(v_j) \in L(v_j)$;

(2) if $v_j v_k \in E$, then $\varphi(v_j) \neq \varphi(v_k)$.

Figure 1 shows a graph G_1 with some list assignment $A_1 = \{\{1, 2, 3\}, \{2, 3\}, \{1, 2\}, \{1, 2, 3\}\}$.

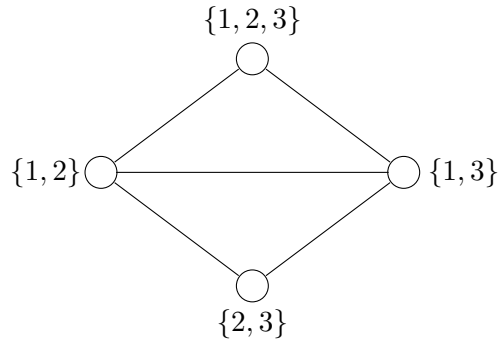


FIGURE 1. Graph G_1 with list assignment A_1 .

Graph G_1 with list assignment A_1 has two proper colorings $S_1 = \{\{2, 1\}, \{1\}, \{3\}, \{2\}\}$ and $S_2 = \{\{3, \}, \{2\}, \{1, \}, \{3\}\}$, which are represented in Figure 2.

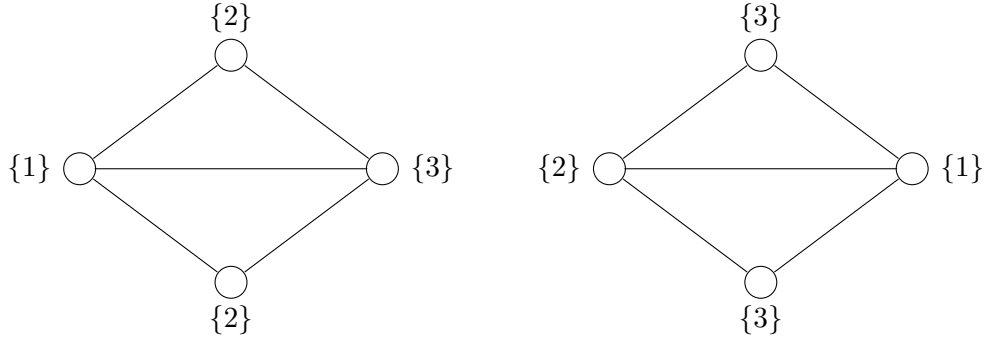
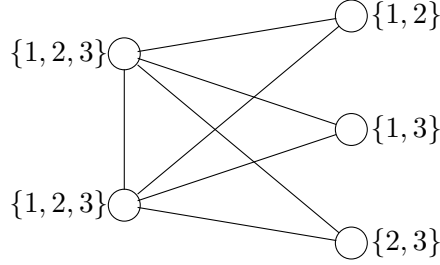


FIGURE 2. Proper colorings S_1 and S_2 of graph G_1 with list assignment A_1 .

Figure 3 shows graph G_2 with list assignment A_2 , but this graph has no proper list coloring.

FIGURE 3. Graph G_2 with list assignment A_2 .

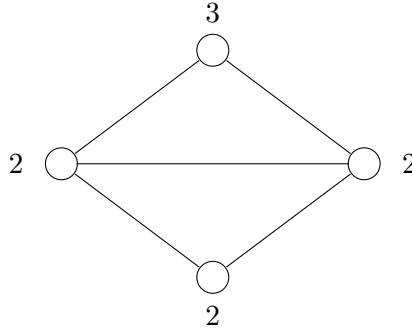
Let f a function from V to the positive integers, that is

$$f : V \rightarrow \{1, 2, \dots\},$$

then a graph G is called *f -choosable* [25] if there is a proper $\mathcal{L}$ -coloring of G whenever $\mathcal{L}$ satisfies

$$|\mathcal{L}| \geq f(v_i) \text{ for all } v_i \in V.$$

For example the graph G_1 represented in Figure 1 is not f -choosable, if f is defined according to Figure 4. In Figure 5 such list assignment A_3 is represented which shows that the graph G_1 is not f -choosable.

FIGURE 4. Function f_1 for graph G_1 .

The most important graph parameter in list coloring is the *choice number* of a graph G denoted by $\chi_l(G)$ and defined as the smallest integer k such that G is $\mathcal{L}$ -colorable for every k -assignment $\mathcal{L}$. The choice number of the graphs represented in Figures 1 and 3 is 3. The $\chi(G)$ chromatic number of these graphs is also 3. It is known that $\chi(G) \leq \chi_l(G)$ for all G [25, 77].

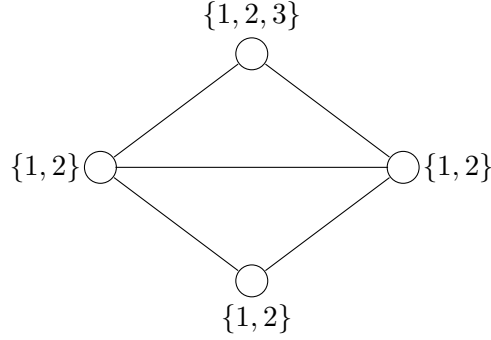


FIGURE 5. List assignment A_3 showing that graph G_1 is not f_1 -choosable.

The exact characterization of the graphs for which the chromatic number is equal with the choice number is an open problem. The graph G_4 represented in Figure 6 represented with a list assignment $A_4 = \{\{1, 3\}, \{2, 3\}, \{1, 2\}, \{1, 2\}, \{2, 3\}, \{1, 3\}\}$ shows $\chi_l(G_4) = 3$, while $\chi(G_4) = 2$.

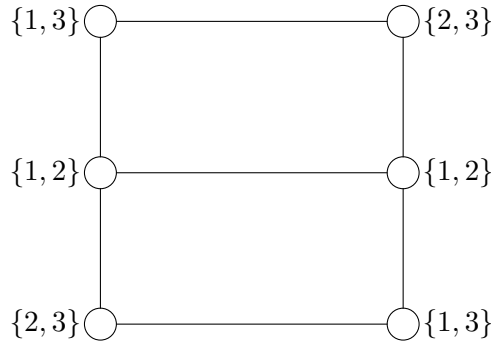


FIGURE 6. Graph G_4 with list assignment A_4 .

2. EXTENSION OF HALL THEOREM

The Hall theorem [41] results the following necessary and sufficient condition of the $\mathcal{L}$ -colorability of K_n (the complete graph on n vertices): K_n is $\mathcal{L}$ -colorable if and only if the union of any k ($1 \leq k \leq n$) lists contain at least k elements.

The classical Hall theorem has several new proofs [4, 6, 10, 61, 62, 66, 72] and many extensions [1, 2, 21, 20, 22, 25, 34, 35, 42, 44, 46, 47, 59, 60, 64, 79].

In 1998 Fred Galvin [36] proposed the following problem as *Problem 1071* in *The American Mathematical Monthly*.

Let G be a finite, undirected, simple graph with vertex set V . Let $C = \{C_x : x \in V\}$ be a family of sets indexed by the vertices of G . For $X \subseteq V$, let $C_X = \cup_{x \in X} C_x$. A set $X \subseteq V$ is C -colorable if one can assign to each vertex $x \in X$ a "color" $c_x \in C_x$ so that $c_x \neq c_y$ whenever x and y are adjacent in G .

a) Prove that if $|C_X| \geq |X|$ whenever X induces a connected subgraph of G , then V is C -colorable.

b) Prove that if every proper subset of V is C -colorable and $|C_V| \geq |V|$, then V is C -colorable.

c) For every connected graph G , find a family $C = \{C_x : x \in V\}$ showing that the condition $|C_V| \geq |V|$ in part **b)** cannot be weakened to $|C_V| \geq |V| - 1$.

Part **a)** was solved by Stephen C. Locke (Florida Atlantic University, Boca Raton, FL), part **b)** by Sung Soo Kim (Hanyang University, Ansan, Kyunggi, Korea), and part **c)** by David Callan (University of Wisconsin, Madison, WI) in 2001 [37, 57].

These assertions on list coloring of graphs can be strengthened in two directions:

- there is a weaker sufficient condition than in part **a)**;
- the condition in part **a)** implies a stronger consequence for which the given condition is necessary and sufficient.

Let $\Gamma(x) = \{y : x \text{ and } y \text{ are adjacent in } G\}$ and $\deg(x) = |\Gamma(x)|$. A set $X \subseteq V$ is P -colorable (path-colorable) if one can assign to each vertex $x \in X$ a color $c_x \in C_x$ so that $c_x \neq c_y$ whenever x and y are connected with a path in G .

d) Prove that if $\deg(v) > |C_v|$ for each $v \in V$, then G is C -colorable.

e) Prove that G is P -colorable iff $|C_X| \geq |X|$ whenever X induces a connected subgraph of G .

Solution to part d)

Let $V = (v_1, v_2, \dots, v_{|V|})$. Then we can assign colors to vertices e.g. in the increasing order of their indices, since a preceding vertex can exclude at most the colors of $\Gamma(v_i)$ adjacent vertices, but the degree of v_i is greater than the number of its adjacent vertices.

Solution to part e)

If G is P -colorable and X induces a connected subgraph of G , then the elements of X are connected with a path in G , therefore $|C_X| \geq |X|$ holds due to the definition of D -colorability.

If $|C_X| \geq |X|$ whenever X induces a connected subgraph S_X of G , then S_X is D -colorable due to Hall's theorem and P -colorability of the connected subgraphs of G implies the D -colorability of G .

We remark that if G is connected, then part **e)** is equivalent with Hall's theorem [41].

3. LIST COLORING OF LATIN AND SUDOKU GRAPHS

For a positive integer $n \geq 2$ the n -order Latin graph $\lambda_n = (V_n, E_n)$ corresponds to an n -order Latin square [23, 24] and is the finite simple graph with $V_n = \{v_{ij} : 1 \leq i, j \leq n\}$, and $E_n = \{(v_{ab}, v_{cd}) : a = c \text{ and } b \neq d, \text{ or } b = d \text{ and } a \neq c\}$.

Figure 9 shows the $n^2 = 16$ vertices of λ_4 . λ_4 has $n \times n \times (n - 1) = 48$ edges:

$$E_4 = \{v_{11}v_{12}, v_{11}v_{13}, v_{11}v_{14}, v_{12}v_{13}, v_{12}v_{14}, v_{13}v_{14}, \dots, v_{43}v_{44}, v_{11}v_{21}, \dots, v_{34}v_{44}\}.$$

v_{11}	v_{12}	v_{13}	v_{14}
v_{21}	v_{22}	v_{23}	v_{24}
v_{31}	v_{32}	v_{33}	v_{34}
v_{41}	v_{42}	v_{43}	v_{44}

FIGURE 7. The vertices of λ_4 .

For a positive integer $m \geq 2$ the m -order Sudoku graph $\sigma_m = (V_m, E_m)$ [43, 68] corresponds to an m -order Sudoku square [14, 15, ?, 30, 33, 54, 55], which is an m^2 -order Latin square divided into $m \times m$ disjoint subsquares of size $m \times m$ called *blocks*. So σ_m is the finite simple graph with $V_m = \{v_{ij} : 1 \leq i, j \leq m^2\}$, and $E_m = \{(v_{ab}, v_{cd}) : a = c \text{ and } b \neq d, \text{ or } b = d \text{ and } a \neq c \text{ or } \lfloor (a+1)/m \rfloor = \lfloor (c+1)/m \rfloor, \text{ and } \lfloor (b+1)/m \rfloor = \lfloor (d+1)/m \rfloor \text{ and } (a, b) \neq (c, d)\}$.

Figure 8 shows the $9^2 = 81$ vertices of σ_3 . σ_3 has $m^2 \times m^2 \times (m^2 - 1) = 648$ edges in the rows and columns, plus $m^2 \times m^2 \times (m - 1)^2 / 2 = 162$ edges in the blocks, so $|E_3| = 810$ as follows:

$$\begin{aligned} E_3 = \{ & v_{11}v_{12}, \dots, v_{18}v_{19}, \dots, v_{91}v_{92}, \dots, v_{98}v_{99}, \\ & v_{11}v_{21}, \dots, v_{81}v_{91}, \dots, v_{19}v_{29}, \dots, v_{89}v_{99}, \\ & v_{11}v_{22}, \dots, v_{23}v_{32}, \dots, v_{77}v_{88}, \dots, v_{89}v_{98} \}. \end{aligned}$$

The following algorithm [23, 24]—described using the pseudocode of [18]—produces a list coloring of an n -order Latin square λ_n for the case $L(v_{ij}) =$

v_{11}	v_{12}	v_{13}	v_{14}	v_{15}	v_{16}	v_{17}	v_{18}	v_{19}
v_{21}	v_{22}	v_{23}	v_{24}	v_{25}	v_{26}	v_{27}	v_{28}	v_{29}
v_{31}	v_{32}	v_{33}	v_{34}	v_{35}	v_{36}	v_{37}	v_{38}	v_{39}
v_{41}	v_{42}	v_{43}	v_{44}	v_{45}	v_{46}	v_{47}	v_{48}	v_{49}
v_{51}	v_{52}	v_{53}	v_{54}	v_{55}	v_{56}	v_{57}	v_{58}	v_{59}
v_{61}	v_{62}	v_{63}	v_{64}	v_{65}	v_{66}	v_{67}	v_{68}	v_{69}
v_{71}	v_{72}	v_{73}	v_{74}	v_{75}	v_{76}	v_{77}	v_{78}	v_{79}
v_{81}	v_{82}	v_{83}	v_{84}	v_{85}	v_{86}	v_{87}	v_{88}	v_{89}
v_{91}	v_{92}	v_{93}	v_{94}	v_{95}	v_{96}	v_{97}	v_{98}	v_{99}

FIGURE 8. The vertices of σ_3 ,

$\{1, 2, \dots, n\}$ for all v_{ij} . The input parameter is n , and the output is a matrix $M_\lambda = [m_{ij}]_{n \times n}$ of the colors of the vertices of the graph λ_n .

```

LATIN( $n$ )
01 for  $i \leftarrow 1$  to  $n$ 
02   do for  $j \leftarrow 1$  to  $n$ 
03     do  $M_\lambda[i, j] \leftarrow i + j - 1 \bmod n$ 
04 return  $M_\lambda$ 

```

The order of the running time of this algorithm corresponds to the number of elements of the matrix M_λ , it is $\Theta(n^2)$.

If the input is $n = 4$, then the output of LATIN is the coloring represented in Figure 9.

1	2	3	4
2	3	4	1
3	4	1	2
4	1	2	3

FIGURE 9. An $\mathcal{L}$ -coloring of λ_4 produced by algorithm LATIN.

It is known [69] that λ_4 has 576 different $\mathcal{L}$ -colorings, if all vertex v_{ij} have the same color list $L(v_{ij}) = (1, 2, 3, 4)$, and λ_9 has

$$5\,524\,751\,496\,156\,892\,842\,531\,225\,600 \sim 5.5 \cdot 10^{27}$$

different colorings [69, 78], if all color lists are $L(v_{ij}) = (1, 2, 3, 4, 5, 6, 7, 8, 9)$. These numbers give the number of 4-order, resp. 9-order Latin squares.

The following algorithm [73] produces a list coloring of the m^2 -order Sudoku graph σ_m for the case $L(v_{ij}) = \{1, 2, \dots, m^2\}$ for all v_{ij} . Its basic idea is similar to the idea of algorithm LATIN. The input parameter is m , and the output is the matrix $M_\sigma = [m_{ij}]_{n \times n}$.

```

SUDOKU( $m$ )
01 for  $i \leftarrow 1$  to  $m^2$ 
02   do for  $j \leftarrow 1$  to  $m^2$ 
03     do  $M_\sigma[i, j] \leftarrow i + j - 1 + \lfloor (i - 1)/m \rfloor + m \lfloor (i - 1)/m \rfloor \bmod m^2$ 
04 return  $M_\sigma$ 

```

The running time of this algorithm also corresponds to the number of elements of the matrix M_σ : it is $\Theta(m^4)$.

If the input is $m = 3$, then the output of SUDOKU is the coloring represented in Figure 10.

1	2	3	4	5	6	7	8	9
4	5	6	7	8	9	1	2	3
7	8	9	1	2	3	4	5	6
2	3	4	5	6	7	8	9	1
5	6	7	8	9	1	2	3	4
8	9	1	2	3	4	5	6	7
3	4	5	6	7	8	9	1	2
6	7	8	9	1	2	3	4	5
9	1	2	3	4	5	6	7	8

FIGURE 10. The $\mathcal{L}$ -coloring of σ_3 , produced by algorithm SUDOKU.

It is known [70] that σ_2 has 288 different $\mathcal{L}$ -colorings, if all lists in $\mathcal{L}$ are equal to $\{1, 2, 3, 4\}$. It is also known [29, 67, 70, 75] that σ_3 has

$$6\,670\,903\,752\,021\,072\,936\,960 \sim 6.7 \cdot 10^{21}$$

different $\mathcal{L}$ -colorings, if all lists in $\mathcal{L}$ are equal with $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. These numbers give the number of 4-order, resp. 9-order Sudoku squares.

In the typical Latin and Sudoku puzzles the color of some subset of vertices is fixed (these vertex-color pairs are the input data of the puzzle). The aim is to complete the coloring, that is determine the number of possible complete colorings and to determine the proper colorings.

The first complexity result is due to Colbourn who proved in 1984 [17], that the problem of decision whether the coloring of a partially colored Latin graph can be completed is NP-complete.

Using Colburn's result Yato and Seta proved in 2003 [73], that the Sudoku ASP problem is NP-complete: if we know a proper vertex coloring of a partially colored Sudoku graph, then it is NP-complete problem to decide whether the given graph has another coloring or not.

In 1956 Behrens [7] proposed a more general problem, than the coloring of the Sudoku graph. He divided the set of the n^2 vertices of the n -order Latin graph λ_n into n disjunct subsets of size n , and introduced the additional requirement that the elements of each subset have to be colored by different colors.

In 2007 Cameron asked [13] the complexity of the completion of the coloring of such generalized graphs, and Vaughan proved in 2009 [76] the NP-completeness.

Of course there are special graphs which are colorable by polynomial algorithms. E.g. the books written by Bach, Berthier, Erickson, Gordon, Inkala and Stuart [3, 8, 26, 39, 50, 71] and papers written by Crook [19] and Provan [65] contain numerous polynomial algorithms solving special Sudoku puzzles having unique solution.

A general framework for handling of similar problems as Sudoku puzzles was proposed by Beasley, Brown and Reid [5].

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DEPARTMENT OF COMPUTER ALGEBRA OF EÖTVÖS LORÁND UNIVERSITY
E-mail address: `tony@compalg.inf.elte.hu`

DEPARTMENT OF NUMERICAL ANALYSIS OF EÖTVÖS LORÁND UNIVERSITY
E-mail address: `birka0@gmail.com`

GENERALIZED WALSH-LIKE SYSTEMS

BALÁZS KIRÁLY

ABSTRACT. By generalization of the notion of product system we give a method to construct biorthogonal Walsh-like systems. These systems are generated by two starting collection of functions and by an m -fold map. The discrete biorthogonal expansion interpolates in the set of basic points. The discrete Fourier coefficients with respect to these product systems and the reconstruction of the functions in basic points can be computed by a generalized Fast Fourier Transform. With these systems we can define efficient interpolation algorithm to approximate surfaces.

1. INTRODUCTION

Previously (in [1] and [2]) we gave a general method to construct biorthogonal Walsh-like systems. These systems are important because we can give an efficient interpolation algorithm with them.

The construction in [1] was started from a twofold map $A : X \rightarrow X$, and from a function $r : X \rightarrow \mathbb{C}$, where X is a nonempty set. Denote $A_0(x) := x$, $A_{n+1}(x) := A(A_n(x))$ ($x \in X, n \in \mathbb{N}$). The system $r_n(x) = r(A_n(x))$ was called Rademacher-like system. We have discretized these functions using the set $X_N := \{x \in X \mid A_N(x) = x_0\} = \{x_k^N \mid k = 0, 1, \dots, 2^N - 1\}$, where x_0 is a fixed element of the interval X . The product system

$$w_\ell := \prod_{k=0}^{N-1} r_k^{\ell_k}, \quad \ell = \sum_{k=0}^{N-1} \ell_k \cdot 2^k$$

of the Rademacher-like system was called Walsh-like system. Under certain condition with respect to r we gave a simple way to construct a system which is biorthogonal to the system w_ℓ ($0 \leq \ell < 2^N$) on the discrete set X_N . The construction was based on a special property of the mixed Dirichlet kernel of the product system.

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In this paper we give a generalization of the mentioned construction to get discrete orthogonal and biorthogonal systems. These systems are generated by two starting sets of functions and by an m -fold map. This special construction allows us to use the generalized Fast Fourier Transform algorithms (see [4] and [5]) to compute discrete Fourier coefficients and partial sums. Especially two-dimensional biorthogonal systems of this type can be obtained in this way. We remark that the Dirichlet kernels of the product systems can be used to construct Haar-like orthogonal and biorthogonal systems too.

2. GENERALIZED PRODUCT SYSTEM

First we generalize the notion of the product system. Let us fix the set $X \neq \emptyset$ and the functions $f_n^{(j)} : X \rightarrow \mathbb{C}$ ($0 \leq j < m, 0 \leq n < N$), where $N, m \in \mathbb{N}^* := \{1, 2, \dots\}$ and $m \geq 2$. We will consider the following finite collection of ordered sets of functions

$$(2.1) \quad \mathcal{F}_n = \{f_n^{(0)}, f_n^{(1)}, \dots, f_n^{(m-1)}\}, \quad n = 0, 1, \dots, N-1.$$

Definition 2.1. *The system $\mathcal{F} := \{F_\ell \mid 0 \leq \ell < m^N\}$ defined by*

$$(2.2) \quad F_\ell := \prod_{k=0}^{N-1} f_k^{(\ell_k)}, \quad \ell = \sum_{k=0}^{N-1} \ell_k \cdot m^k, \quad \ell_k \in \{0, 1, \dots, m-1\}$$

*is called the **generalized product system** generated by the collection $(\mathcal{F}_n, n < N)$.*

In the special case when $m = 2$ and $f_n^{(0)} = 1$ and $f_n^{(1)} = f_n$ we reobtain the original definition of the product system.

Obviously the product of the ordered sets of functions $\mathcal{F}_j$ ($0 \leq j < N$) is equal to $\mathcal{F}$:

$$(2.3) \quad \mathcal{F}_0 \cdot \mathcal{F}_1 \cdots \mathcal{F}_{N-1} = \{F_\ell \mid 0 \leq \ell < m^N\} = \mathcal{F}.$$

Let us denote by $\mathcal{G} := \{G_\ell \mid 0 \leq \ell < m^N - 1\}$ the generalized product system of the collection

$$\mathcal{G}_n = \{g_n^{(0)}, g_n^{(1)}, \dots, g_n^{(m-1)}\}, \quad 0 \leq n < N.$$

Then the **mixed Dirichlet kernels of the systems $\mathcal{F}$ and $\mathcal{G}$** are defined by

$$(2.4) \quad D_n(s, t) := \sum_{\ell=0}^{n-1} F_\ell(s) \overline{G_\ell(t)} \quad (n \in \mathbb{N}).$$

The constructions of discrete biorthogonal systems are based on the following claim which can be proved by induction.

Lemma 2.2. *In the case $n = m^N$ the mixed Dirichlet kernel of the product systems can be written in a product form*

$$(2.5) \quad D_{m^N}(s, t) = \sum_{\ell=0}^{m^N-1} \prod_{k=0}^{N-1} f_k^{(\ell_k)}(s) \cdot \bar{g}_k^{(\ell_k)}(t) = \prod_{i=0}^{N-1} \sum_{j=0}^{m-1} f_i^{(j)}(s) \cdot \bar{g}_i^{(j)}(t).$$

This formula is the generalization of the Paley Identity (see [3] and [6]).

To construct biorthogonal product systems we use the notion of m -fold map.

Definition 2.3. *Let X be a nonempty set and $m \in \mathbb{N}^* := \{1, 2, \dots\}$ a fixed number. The function $A : X \rightarrow X$ is called **m -fold map of X** , if for every $y \in X$ there exist exactly m different points $x_0, x_1, \dots, x_{m-1} \in X$, such that $A(x_0) = A(x_1) = \dots = A(x_{m-1}) = y$, i.e. for the preimage of y we have*

$$A^{-1}(y) = \{x_0, x_1, \dots, x_{m-1}\}.$$

Definition 2.4. *The systems $\mathcal{F} := \{f^{(0)}, f^{(1)}, \dots, f^{(m-1)}\}$ and $\mathcal{G} := \{g^{(0)}, g^{(1)}, \dots, g^{(m-1)}\}$ are called **biorthogonal with respect to the m -fold map A of X** , if for every $y \in X$*

$$(2.6) \quad \frac{1}{m} \sum_{x \in A^{-1}(y)} f^{(i)}(x) \bar{g}^{(j)}(x) = \delta_{ij} \quad (0 \leq i, j < m).$$

*In the case $\mathcal{F} = \mathcal{G}$ we say that the system $\mathcal{F}$ is **orthogonal with respect to the m -fold map A** .*

It is easy to see that condition (2.6) is equivalent to

$$(2.7) \quad \frac{1}{m} \sum_{i=0}^{m-1} f^{(i)}(u) \bar{g}^{(i)}(v) = \delta_{uv} \quad (u, v \in A^{-1}(y)).$$

In the case $m = 2$ the following notion is useful.

Definition 2.5. *The function $f^{(0)} : X \rightarrow \mathbb{C}$ is called **even with respect to the 2-fold map A of X** if $A(x') = A(x'')$ implies $f^{(0)}(x') = f^{(0)}(x'')$. If for the function $f^{(1)} : X \rightarrow \mathbb{C}$ in the case $A(x') = A(x'')$, $x' \neq x''$ equality $f^{(1)}(x') = -f^{(1)}(x'')$ is satisfied, then we say that $f^{(1)}$ is **odd with respect to A** .*

It is easy to check the following claim.

Corollary 2.6. *If the function $f^{(0)}$ is even and $f^{(1)}$ is odd with respect to the 2-fold map A and $|f^{(0)}(x)| = |f^{(1)}(x)| = 1$, then the system $\mathcal{F} = \{f^{(0)}, f^{(1)}\}$ is orthogonal with respect to the map A .*

In our construction we need the following iteration of the m -fold map A . Let $A_0(x) := x$ and $A_{n+1}(x) := (A \circ A_n)(x) := A(A_n(x))$ for every $x \in X$. It is clear that A_n is an m^n -fold map of X . To discretize the generalized product system we introduce the set

$$(2.8) \quad X_N := \{x \in X : A_N(x) = x_0\} = \{x_k^N \mid k = 0, 1, \dots, m^N - 1\}$$

and the scalar product on X_N :

$$(2.9) \quad [f, g]_{X_N} := \frac{1}{m^N} \sum_{x \in X_N} f(x) \bar{g}(x) \quad (f, g : X \rightarrow \mathbb{C}).$$

To get biorthogonal product systems we start from the sets of functions $\mathcal{F}$ and $\mathcal{G}$ and introduce the collections of functions

$$(2.10) \quad \mathcal{F}_n := \{f \circ A_n \mid f \in \mathcal{F}\}, \quad \mathcal{G}_n = \{f \circ A_n \mid g \in \mathcal{G}\}$$

and denote by $\mathcal{F} := \{F_\ell, 0 \leq \ell < m^N\}$ and $\mathcal{G} := \{G_\ell, 0 \leq \ell < m^N\}$ the generalized product systems generated by $\{\mathcal{F}_n, 0 \leq n < N\}$ and $\{\mathcal{G}_n, 0 \leq n < N\}$. In this case we say that the product systems $\mathcal{F}$ and $\mathcal{G}$ are generated by the set $\mathcal{F}$ and $\mathcal{G}$ and by the m -fold map A

Theorem 2.7. *Suppose that the systems $\mathcal{F}$ and $\mathcal{G}$ are biorthogonal with respect to m -fold map A of X . Then the product systems $\mathcal{F}$ and $\mathcal{G}$ generated by $\mathcal{F}$ and $\mathcal{G}$ and by A are biorthogonal with respect to the discrete scalar product of the set X_N , i.e.*

$$(2.11) \quad [F_\ell, G_k]_{X_N} = \delta_{\ell k} \quad (0 \leq \ell, k < m^N).$$

Moreover the mixed Dirichlet kernel of the generalized product systems satisfies

$$(2.12) \quad D_{m^N}(s, t) = m^N \delta_{st} \quad (s, t \in X_N)$$

Proof. It is easy to see that (2.11) and (2.12) are equivalent. By (2.5) we have

$$(2.13) \quad D_{m^N}(s, t) = \sum_{\ell=0}^{m^N-1} F_\ell(s) \cdot \bar{G}_\ell(t) = \prod_{k=0}^{N-1} \sum_{i=0}^{m-1} f_k^{(i)}(s) \cdot \bar{g}_k^{(i)}(t).$$

If $s \neq t$, then there exists a first common image y and an index p ($0 \leq p < N$) such that

$$(2.14) \quad y = A_{p+1}(s) = A_{p+1}(t), \quad u := A_p(s) \neq v := A_p(t), \quad u, v \in A^{-1}(y).$$

By the definition of A_p we have

$$f_p^{(i)}(s) = f^{(i)}(A_p(s)) = f^{(i)}(u), \quad g_p^{(i)}(t) = g^{(i)}(A_p(t)) = g^{(i)}(v)$$

and consequently from (2.7) we get

$$\sum_{i=0}^{m-1} f_p^{(i)}(s) \cdot \bar{g}_p^{(i)}(t) = \sum_{i=0}^{m-1} f^{(i)}(u) \cdot \bar{g}^{(i)}(v) = \delta_{uv} = 0.$$

This means that there exists a zero factor in the product of the Dirichlet-kernel, which implies that $D_{m^N}(s, t) = 0$ if $s \neq t$.

If $t = s$ then $u := A_p(s) = v := A_p(t)$ for every $0 \leq p < N$ and consequently by (2.7) every factor in the product is m . $\square$

Applying Theorem 2.7 and Corollary 2.6 we get

Corollary 2.8. *If the conditions of Corollary 2.6 are satisfied for the system $\mathcal{F} = \{f^{(0)}, f^{(1)}\}$ then the product system $\mathcal{F}$ generated by $\mathcal{F}$ and by the twofold map A is orthonormal.*

Denote

$$S_n f := \sum_{k=0}^{n-1} [f, G_k]_{X_N} F_k \quad (0 \leq n \leq m^N)$$

the partial sums of the biorthogonal expansion of $f : X \rightarrow \mathbb{C}$ with respect the systems $\mathcal{F}$ and $\mathcal{G}$ defined in Theorem 2.7. Then from (2.12) it follows

Corollary 2.9. *The partial sum $S_{m^N} f$ interpolate the function f at the point of X_N :*

$$(S_{m^N} f)(x) = f(x) \quad (x \in X_N).$$

3. KRONECKER-PRODUCTS OF PRODUCT SYSTEMS

For functions $A_1, A_2 : X \rightarrow X$ we introduce the Kronecker product $X^2 := X \times X$ and the map $A : X^2 \rightarrow X^2$ defined by

$$(3.1) \quad A(x) := (A_1 \diamond A_2)(x) := (A_1(x_1), A_2(x_2)) \quad (x = (x_1, x_2) \in X^2).$$

In this case the preimage of any $y = (y_1, y_2) \in X^2$ can be written in the form

$$A^{-1}(y) = \{(x_1, x_2) \in X^2 \mid A_1(x_1) = y_1, A_2(x_2) = y_2\} = A_1^{-1}(y_1) \times A_2^{-1}(y_2)$$

Thus if A_1 and A_2 are m -fold map of the set X then A is a m^2 -fold map of X^2 .

The **Kronecker product of the functions** $f, g : X \rightarrow \mathbb{C}$ is denoted by

$$(f \times g)(x) := f(x_1)g(x_2) \quad (x = (x_1, x_2) \in X^2)$$

and the **Kronecker product of the set of functions** $\mathcal{F}$ and $\mathcal{G}$ is defined by

$$\mathcal{F} \times \mathcal{G} := \{f \times g \mid f \in \mathcal{F}, g \in \mathcal{G}\}.$$

Let

$$\mathcal{F} := \mathcal{F}_0 \cdot \mathcal{F}_1 \cdots \mathcal{F}_{N-1}, \quad \mathcal{G} := \mathcal{G}_0 \cdot \mathcal{G}_1 \cdots \mathcal{G}_{N-1}$$

denote the product system generated by the systems

$$\mathcal{F}_k := \{f_k^{(\ell)} : 0 \leq \ell < m\}, \quad \mathcal{G}_k := \{g_k^{(\ell)} : 0 \leq \ell < m\} \quad (0 \leq k < N).$$

Then

$$\mathcal{F} \times \mathcal{G} = \left(\prod_{k=0}^{N-1} \mathcal{F}_k \right) \times \left(\prod_{k=0}^{N-1} \mathcal{G}_k \right) = \prod_{k=0}^{N-1} (\mathcal{F}_k \times \mathcal{G}_k).$$

Remark 3.1. *This means that the Kronecker-product of product systems is the product system of the Kronecker-product of the generating systems.*

Property (2.6) with respect to map A_1 and A_2 inherits to the map $A = A_1 \diamond A_2$. Indeed suppose that $f_1 \in \mathcal{F}$, $g_1 \in \mathcal{G}$ are biorthogonal with respect to A_1 and $f_2 \in \mathcal{F}$, $g_2 \in \mathcal{G}$ are biorthogonal with respect to A_2 . Then for the Kronecker products $f := f_1 \times f_2$ and $g := g_1 \times g_2$ and for $y = (y_1, y_2)$ we get

$$\begin{aligned} \frac{1}{m^2} \sum_{x \in A^{-1}(y)} f(x) \bar{g}(x) &= \frac{1}{m^2} \sum_{x \in A_1^{-1}(y_1) \times A_2^{-1}(y_2)} f_1(x_1) f_2(x_2) \bar{g}_1(x_1) \bar{g}_2(x_2) = \\ &= \frac{1}{m} \sum_{x_1 \in A_1^{-1}(y_1)} f_1(x_1) \bar{g}_1(x_1) \cdot \frac{1}{m} \sum_{x_2 \in A_2^{-1}(y_2)} f_2(x_2) \bar{g}_2(x_2) = \delta_{f_1 g_1} \delta_{f_2 g_2}. \end{aligned}$$

Thus the following claim is true.

Corollary 3.2. *If f_1 and g_1 are biorthogonal with respect to A_1 and if f_2 and g_2 are biorthogonal with respect to A_2 then the Kronecker products $f_1 \times f_2$ and $g_1 \times g_2$ are biorthogonal with respect to the map $A = A_1 \diamond A_2$.*

Combining these results we get a general method to construct two dimensional biorthogonal product systems.

Theorem 3.3. *Suppose that the systems $\mathcal{F}$ and $\mathcal{G}$ are biorthogonal with respect to m -fold map A of X . Then the Kronecker product $\mathcal{F}^2 := \mathcal{F} \times \mathcal{F}$ and $\mathcal{G}^2 := \mathcal{G} \times \mathcal{G}$ are biorthogonal with respect to the m^2 -fold map $A^2 := A \diamond A$ of X^2 . Thus the product systems generated $\mathcal{F}^2$ and $\mathcal{G}^2$ are biorthogonal.*

4. EXAMPLES

1. Let $\mathcal{I} := [0, 1)$. And let $\xi \in (0, 1)$. Let $\mathcal{A}_1 : [0, \xi) \rightarrow [0, 1)$ be a strictly monoton function, and $\mathcal{A}_2 : [\xi, 1) \rightarrow [0, 1)$ be another strictly monoton function. Let $[0, 1)$ be the range of maps $\mathcal{A}_1$ and $\mathcal{A}_2$. It is easy to see, that

$$(4.1) \quad A(x) := \begin{cases} \mathcal{A}_1(x) & 0 \leq x < \xi, \\ \mathcal{A}_2(x) & \xi \leq x < 1. \end{cases}$$

is a two-fold map of interval $\mathcal{I}$.

We can easily construct odd and even function with respect to this two-fold map. Let $f^{(0)}, f^{(1)} : \mathcal{I} \rightarrow \mathbb{R}$ such that $f^{(0)}(x) := 1$ for every $x \in \mathcal{I}$ and

$$f^{(1)}(x) := \begin{cases} 1 & 0 \leq x < \xi, \\ -1 & \xi \leq x < 1. \end{cases}$$

It is easy to see that $f^{(0)}$ is an even and $f^{(1)}$ is an odd function with respect to the twofold map A . Because the function $\mathcal{F} = \{f^{(0)}, f^{(1)}\}$ satisfies the conditions of the Corollary 2.6, the function $\mathcal{F}$ is orthogonal with respect to the map A and the product system $\mathcal{F}$ generated by $\mathcal{F}$ and by the twofold map is orthonormal.

Remark 4.1. *If $\xi = \frac{1}{2}$, then $(\mathcal{F}_n, 0 \leq n < N)$ is the original Rademacher-system and $(\mathcal{F}_m, 0 \leq m < 2^N)$ is the original Walsh-Paley-system.*

2. Let A be the map which is defined in 4.1, then $\mathcal{A} := A \diamond A$ is a 4-fold map on the unit square. Let us denote by $\mathcal{F}^2$ The Kronecker product of the set of functions $\mathcal{F}$, which is presented in example 1 with itself, ie.

$$\mathcal{F}^2(x, y) := (\mathcal{F} \times \mathcal{F})(x, y) = \{1, f^{(1)}(x), f^{(1)}(y), f^{(1)}(x) \cdot f^{(1)}(y)\} \quad x, y \in [0, 1)$$

Because $\mathcal{F}^2$ satisfies the conditions of theorem 2.7, the product system $\mathcal{F}^2$ of system $\mathcal{F}^2$ is orthonormal.

Remark 4.2. *If $\xi = \frac{1}{2}$, then $(\mathcal{F}_n^2, 0 \leq n < N^2)$ is the two-dimensional Rademacher-system and system $(\mathcal{F}_k^2, 0 \leq k < 4^{N^2})$ is the two-dimensional Walsh-Paley-system.*

5. FURTHER OPPORTUNITIES

By the connection between the original Walsh and Haar systems we can define Generalized Haar-like Systems. With these systems we will be able to generalize the method of Fast Fourier Transform. By this we will get an efficient, quickly computable algorithm for image processing and storage.

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UNIVERSITY OF PÉCS, FACULTY OF SCIENCES, INSTITUTE OF MATHEMATICS AND INFORMATICS

E-mail address: kiralyb@gamma.ttk.pte.hu

CONSTRUCTING ORTHOGONAL SYSTEMS USING BLASCHKE PRODUCTS

LEVENTE LÓCSI

ABSTRACT. The Blaschke functions play an important role in several areas of mathematics and informatics e.g. signal processing and hyperbolic geometry. Rational complex orthogonal systems in the Hardy space $\mathcal{H}^2$ such as the Malmquist–Takenaka systems can be written in an explicit form using Blaschke functions and their products.

In this paper we analyse the $D(z, \zeta)$ Dirichlet kernel of these orthogonal systems, give an explicit form for the values $D(z, z)$ in special cases, and construct new orthogonal systems by observing $D(z, \zeta_k)$ slices of the Dirichlet kernel with properly chosen ζ_k values in $[0, 2\pi)$. Discrete orthogonality properties shall also be stated.

1. INTRODUCTION

In this section we introduce our functions of interest, define the Malmquist–Takenaka systems which are rational complex functions and form an orthogonal system, and recall the basic properties of the Dirichlet kernel in the case of trigonometric basis functions.

1.1. Malmquist–Takenaka systems. Denote by $\mathbb{C}$ the set of complex numbers and let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk, defined as $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ the unit circle and $\mathbb{D}^* := \mathbb{C} \setminus (\mathbb{D} \cup \mathbb{T})$. The natural numbers will be considered as the set $\mathbb{N} := \{1, 2, 3, \dots\}$.

We will consider functions of the $\mathcal{A}$ disk algebra, i.e. the set of functions analytic on $\mathbb{D} \cup \mathbb{T}$. The scalar product on $\mathbb{T}$ is defined by

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) \overline{g}(e^{it}) dt.$$

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We shall examine functions generated by the collection of

$$\varphi_k(z) := \frac{1}{(1 - \overline{a_k}z)^{m_k}} \quad (z \in \mathbb{C}; n \in \mathbb{N}; k = 1, 2, \dots, n),$$

where $a_k \in \mathbb{D}$ ($k = 1, 2, \dots, n$) and $m_k = \sum_{i \leq k, a_i = a_k} 1$ the multiplicity of the parameter a_k . We note that φ_k has a pole in $a_k^* = 1/\overline{a_k} \in \mathbb{D}^*$ and $\Phi := (\varphi_k : k = 1, 2, \dots, n) \subset \mathcal{A}$.

By applying the Gram–Schmidt orthogonalization procedure to Φ we obtain an orthonormal system $\Psi := (\psi_k : k = 1, 2, \dots, n)$ on $\mathbb{T}$ the so-called Malmquist–Takenaka system (introduced in [4] and [6], see also [3]) which can be expressed by the Blaschke functions

$$(1) \quad B_a(z) := \frac{z - a}{1 - \overline{a}z} \quad (a \in \mathbb{D}; z \in \mathbb{C}).$$

Namely

$$(2) \quad \psi_k(z) = \frac{\sqrt{1 - |a_k|^2}}{1 - \overline{a_k}z} \prod_{j=1}^{k-1} B_{a_j}(z) \quad (z \in \mathbb{C}; k = 1, 2, \dots, n),$$

which suggests a convenient computation method for the values of ψ_k . For further examinations related to the Malmquist–Takenaka system and Blaschke product systems see e.g. [2, 5].

This system serves as a starting point in our research: we shall examine the Dirichlet kernel of this orthogonal system.

1.2. The trigonometric Dirichlet kernel. Now, let us recall the definition and basic properties of the (original) trigonometric Dirichlet kernel.

Consider the complex trigonometric system: e^{inx} ($n \in \mathbb{Z}$). (The real and imaginary parts of which correspond to the real trigonometric system: $1, \cos x, \sin x, \cos 2x, \sin 2x, \dots$)

Then the n th degree Fourier series approximation of an $f \in C_{2\pi}$ (2π periodic and continuous) function can be written as:

$$(3) \quad (S_n f)(x) = \sum_{k=-n}^n \left(\frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-ikt} dt \right) e^{ikx} = \frac{1}{2\pi} \int_0^{2\pi} f(t) \underbrace{\left(\sum_{k=-n}^n e^{ikx} e^{-ikt} \right)}_{D_n(x-t)=D_n(x,t)} dt.$$

We have written the linear combination of the basis functions with the coefficients given as the scalar product (integral) of f and the appropriate basis function. And so the Fourier series approximation can be written as an integral transformation of f with the kernel function underbraced and denoted as $D_n(x - t) = D_n(x, t)$.

Note that in this case the Dirichlet kernel can be defined as a function of either one or two variables, because of its following property: $D_n(x + \nu, t + \nu) = D_n(x, t)$ ($\nu \in \mathbb{R}$), so basically only the difference of x and t matters. Since D_n is also 2π periodic, we shall examine values where $x, t \in [0, 2\pi)$.

The trigonometric Dirichlet kernel can be written in the explicit form:

$$D_n(x) = \frac{\sin((2n+1)x/2)}{\sin x/2} \quad (x \neq 0), \quad D_n(0) = 2n+1.$$

2. THE DIRICHLET KERNEL OF MALMQUIST–TAKENAKA SYSTEMS

In this section we will show a generalized form of the Dirichlet kernel, and state some properties in the case of Malmquist–Takenaka systems, and examine further special cases.

For a general form of the Dirichlet kernel, consider an arbitrary $(\phi_i : i \in \mathbb{N})$ orthogonal system. We shall be thinking of the Hardy space $\mathcal{H}^2$ (the space of functions square integrable on the complex unit circle $\mathbb{T}$).

The n th degree Fourier series approximation of $f \in \mathcal{H}^2$ can be written (analogously to (3)) as:

$$(4) \quad (S_n f)(x) = \sum_{k=1}^n \left(\frac{1}{2\pi} \int_0^{2\pi} f(t) \overline{\phi_k}(t) dt \right) \phi_k(x) = \frac{1}{2\pi} \int_0^{2\pi} f(t) \underbrace{\left(\sum_{k=1}^n \phi_k(x) \overline{\phi_k}(t) \right)}_{D_n(x,t)} dt.$$

Thus the general Dirichlet kernel is defined as:

$$(5) \quad D_n(z, \zeta) = \sum_{k=1}^n \phi_k(z) \overline{\phi_k}(\zeta) \quad (z, \zeta \in \mathbb{T}),$$

with the symbols $z, \zeta \in \mathbb{T}$ used, which correspond to $x, t \in [0, 2\pi)$.

Since the Malmquist–Takenaka systems form an orthonormal system in $\mathcal{H}^2$, the Dirichlet kernel can be calculated, and can be expressed using Blaschke products.

Theorem 1. *With given $n \in \mathbb{N}$ and $a_k \in \mathbb{D}$ ($k = 1, 2, \dots, n$) the Dirichlet kernel of the Malmquist–Takenaka system ψ_k ($k = 1, 2, \dots, n$) can be written as:*

$$(6) \quad D_n(z, \zeta) = \sum_{k=1}^n \psi_k(z) \overline{\psi_k(\zeta)} = \frac{\prod_{k=1}^n B_{a_k}(z) \overline{B_{a_k}(\zeta)} - 1}{z \bar{\zeta} - 1} \quad (z \neq \zeta),$$

$$(7) \quad D_n(z, z) = \sum_{k=1}^n \psi_k(z) \overline{\psi_k(z)} = \sum_{k=1}^n \frac{1 - |a_k|^2}{|1 - \overline{a_k} z|^2} \quad (z = \zeta).$$

Proof. The formulas (6) and (7) can be derived by induction after substituting the expressions (1) and (2) in (5). We do not provide detailed calculations here. $\square$

In order to get translation invariant systems and attempt to find wavelet-like constructions a special distribution of the a_k parameters may be considered. In this case a nice relation can be found between the $D_n(t, t)$ values and the Poisson kernel function stated in Theorem 2.

Recall the Poisson kernel for a given $0 \leq r < 1$:

$$P_r(t) = \frac{1 - r^2}{|1 - r \cdot \exp it|^2} = \sum_{l \in \mathbb{Z}} r^{|l|} \exp i l t = \frac{1 - r^2}{1 - 2r \cos t + r^2}.$$

Theorem 2. *If $n \in \mathbb{N}$ and $a_k = r \cdot \exp i \frac{2k\pi}{n}$, then considering the Dirichlet kernel of the Malmquist–Takenaka system ψ_k ($k = 0, 1, \dots, n-1$; $0 \leq r < 1$) the following identity holds:*

$$D_n(t, t) = \sum_{l \in \mathbb{Z}} r^{|nl|} \exp i n l t = \frac{1 - r^{2n}}{1 - 2r^n \cos nt + r^{2n}},$$

which is basically a dilated transform of the Poisson kernel by the factor n , and with parameter r^n .

Figure 1 shows the Dirichlet kernel of the Malmquist–Takenaka system defined by the parameters $a_1 = 1/2$ and $a_2 = -1/2$ on the two dimensional interval $[0, 2\pi] \times [0, 2\pi]$. The central $D_2(t, t)$ values are also highlighted: the plot of the appropriate Poisson kernel fits.

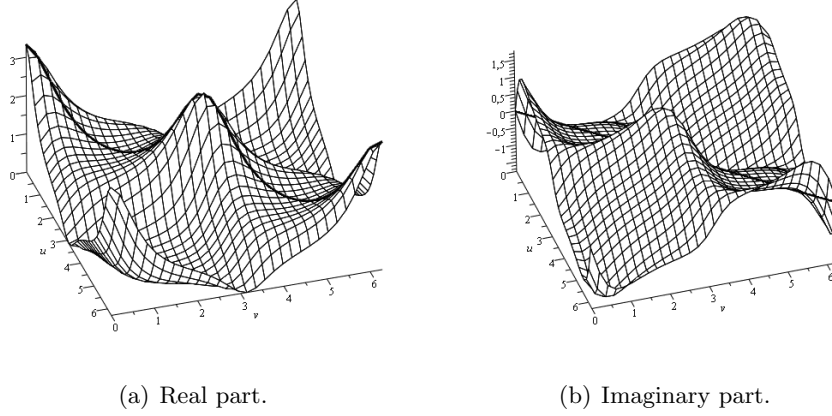


FIGURE 1. The Dirichlet kernel of a Malmquist–Takenaka system.

3. CONSTRUCTING NEW ORTHOGONAL SYSTEMS

In this section we describe our method to find new orthogonal systems using the Dirichlet kernel of Malmquist–Takenaka systems. The functions of these systems can be expressed by terms of Blaschke products, which also play an important role throughout the construction.

These systems depend on many parameters which can be arbitrarily chosen (the formerly mentioned $a_k \in \mathbb{D}$ parameters of the rational complex functions, and a further $\epsilon \in \mathbb{T}$ to be introduced here), and therefore we can flexibly adapt the system to a problem at hand.

Lemma 3. *With given $n \in \mathbb{N}$, $a_k \in \mathbb{D}$ ($k = 1, 2, \dots, n$) and $\epsilon \in \mathbb{T}$ the equality*

$$(8) \quad \prod_{k=1}^n B_{a_k}(z) = \epsilon$$

with a Blaschke product on the left-hand side has exactly n distinct solutions: $z = \zeta_1, \zeta_2, \dots, \zeta_n \in \mathbb{T}$.

And these roots have the following property:

$$(9) \quad D_n(\zeta_k, \zeta_l) = \delta_{k,l} \cdot D_n(\zeta_k, \zeta_k),$$

where D_n is the Dirichlet kernel at hand, and $\delta_{k,l}$ is the Kronecker symbol.

Proof. Assertion (8) can be deduced from the fact that an n factor Blaschke product is an n -fold map of $\mathbb{T}$ and can be confirmed using the argument

function of Blaschke products (see [1, 5]). And statement (9) can be confirmed considering (6) and (8). $\square$

Equipped with the previous lemma now we shall observe some 'slices' of the $D_n(z, \zeta)$ Dirichlet kernel, and state both the continuous and discrete orthogonality of the produced system.

Definition 4. For given $n \in \mathbb{N}$, $a_k \in \mathbb{D}$ ($k = 1, 2, \dots, n$), $\epsilon \in \mathbb{T}$ we may find the ζ_k ($k = 1, 2, \dots, n$) values as stated in Lemma 3. Let us observe the following 'slices' of the associated Dirichlet kernel:

$$(10) \quad H_k(z) := D_n(z, \zeta_k) \quad (k = 1, 2, \dots, n).$$

Recall that the scalar product in the continuous case is considered as

$$\langle F, G \rangle = \frac{1}{2\pi} \int_0^{2\pi} F(t) \overline{G(t)} dt,$$

and let us define the discrete scalar product in our context with weight function ϱ as follows:

$$[F, G]_n = \sum_{k=1}^n F(\zeta_k) \cdot \overline{G(\zeta_k)} \cdot \varrho(\zeta_k), \quad \varrho(\zeta_k) = \frac{1}{D_n(\zeta_k, \zeta_k)}.$$

Theorem 5. The H_k ($k = 1, 2, \dots, n$) functions in Definition 4 form an orthogonal system with respect both to the continuous and the discrete scalar product, i.e.:

$$\begin{aligned} (1) \quad \langle H_k, H_l \rangle &= \delta_{k,l} \cdot D_n(\zeta_k, \zeta_k) & (k, l = 1, 2, \dots, n) \\ (2) \quad [H_k, H_l]_n &= \delta_{k,l} \cdot D_n(\zeta_k, \zeta_k) & (k, l = 1, 2, \dots, n) \end{aligned}$$

Proof. The continuous orthogonality is the consequence of the orthogonality of the Malmquist–Takenaka system, and the discrete orthogonality can be seen by observing the $D_n(\zeta_k, \zeta_l)$ values described in (9). $\square$

Corollary 6. Note that Theorem 5 also gives a quadrature formula in the case of functions in $\mathcal{R}_n \overline{\mathcal{R}_n}$ where $\mathcal{R}_n = \text{span} \{ \varphi_k : k = 1, 2, \dots, n \}$.

Proof. Theorem 5 stated that $\langle H_k, H_l \rangle = [H_k, H_l]_n$ and the functions φ_k , ψ_k and H_k ($k = 1, 2, \dots, n$) all span the same subspace of $\mathcal{H}^2$. $\square$

4. SUMMARY

In this paper we have investigated the D_n Dirichlet kernel of Malmquist–Takenaka systems, and found that these are also expressible using Blaschke products.

We have stated a relation between the Poisson kernel and the $D_n(z, z)$ values in special cases.

And furthermore we have constructed new orthogonal systems by observing $D_n(z, \zeta_k)$ slices of the Dirichlet kernel. These orthogonal systems depend on many parameters and therefore can be flexibly adapted to different problems.

5. FURTHER RESEARCH

In our further research we would like to observe the behaviour of the Dirichlet kernel and the associated orthogonal systems in many concrete cases from simple to more difficult examples, both with uniformly and non-uniformly distributed parameters of the rational system.

With carefully chosen parameters we may also find wavelet-like constructions, and one shall examine the multiresolution-like properties of those.

And finally the possible application areas in signal processing are also to be explored.

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DEPARTMENT OF NUMERICAL ANALYSIS, FACULTY OF INFORMATICS, EÖTVÖS LORÁND
UNIVERSITY

E-mail address: locsi@inf.elte.hu

ASYMPTOTIC STABILITY FOR CELL DYNAMICS AFTER BONE MARROW TRANSPLANTATION

RADU PRECUP, MARCEL-ADRIAN ȘERBAN, AND DAMIAN TRIF

ABSTRACT. The basic idea of the stem cell transplantation consists in adding in competition with the host cells a new population of donor cells. A new equation is added to a previous system essentially owed to Dingli and Michor, which is itself modified in order to incorporate new parameters standing for the intensity of anti-graft, anti-host and anti-leukemia effects. The aim of this paper is to find the equilibria of this extended system, to study their stability and to find the boundary of the attraction basins of the stable equilibria. Given the initial concentrations of the host cells, we may calculate an initial concentration of the donor cells so that, in time, host cells will be eliminated and completely replaced by donor cells, guaranteeing the elimination of leukemia.

1. INTRODUCTION

In [2] a simple mathematical model essentially owed to Dingli and Michor [3] was used in order to characterize normal and leukemic states and to explain basic pathways through which the robustness of the hematopoietic system can fail leading to leukemia. Assume that at each time t , the cell population divides into two: the normal population $x(t)$ and the leukemic population $y(t)$ with $x(0) = x_0$, $y(0) = y_0$. Denote by a, b, c and A, B, C (model parameters) the intrinsic (i.e., in the absence of any constraints) growth, microenvironment sensitivity and death rates of normal and leukemic cells, respectively. The conservation laws for normal and leukemic cells are expressed as a system of

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two differential equations:

$$(1) \quad \begin{cases} x' = \frac{a}{1+b(x+y)}x - cx \\ y' = \frac{A}{1+B(x+y)}y - Cy. \end{cases}$$

We assume that $a > c$, $A > C$ and denote

$$d := \frac{1}{b} \left(\frac{a}{c} - 1 \right), \quad D := \frac{1}{B} \left(\frac{A}{C} - 1 \right).$$

The stability analysis in [2] shows that the zero solution $(0, 0)$ is always unstable. If $d > D$, then $(d, 0)$ is the unique asymptotically stable equilibrium and if $d < D$, then $(0, D)$ is the unique asymptotically stable equilibrium. These happen independently from the initial concentrations $x_0 > 0$, $y_0 > 0$. Thus we may say that the normal hematopoietic state is characterized by the inequality $d > D$ while the leukemic hematopoietic state corresponds to the inequality $d < D$.

The basic idea of the stem cell transplantation consists in adding, say at time $t = 0$, in competition with x_0, y_0 (host cells) a new population (donor cells) z_0 . If the aggressiveness of z against x, y (graft-versus-host and graft-versus-leukemia) compensates that of the x and y against z (anti-graft effect), and if the initial concentrations x_0, y_0 are smaller enough as compared with z_0 , then, in time, host cells are eliminated and completely replaced by donor cells guaranteeing the elimination of cancer.

Mathematically, this means that a new equation in z is added to the previous system in x and y which is itself modified in order to incorporate the new competition (mutual "aggressiveness") between z , on one side, and x and y , on the other. Assuming that intrinsic growth, sensitivity and death rates of the donor cell population are those of the normal host cell population (human invariant kinetic parameters), namely a, b, c , in [1] it was proposed the

following model for the cellular dynamics after bone marrow transplantation:

$$(2) \quad \begin{cases} x' = \frac{a}{1 + b(x + y + z)} \frac{x + y}{x + y + gz} x - cx \\ y' = \frac{A}{1 + B(x + y + z)} \frac{x + y}{x + y + Gz} y - Cy \\ z' = \frac{a}{1 + b(x + y + z)} \frac{z}{z + h(x + y)} z - cz. \end{cases}$$

Here the growth inhibitory factors

$$\frac{1}{1 + g\frac{z}{x+y}}, \quad \frac{1}{1 + G\frac{z}{x+y}}, \quad \frac{1}{1 + h\frac{x+y}{z}}$$

take into account the cell-cell interactions, quantitatively by ratios $\frac{z}{x+y}$ and $\frac{x+y}{z}$, and qualitatively by parameters h, g, G standing for the intensity of anti-graft, anti-host and anti-leukemia effects, respectively.

Numerical simulations performed in [1] for the leukemic case $d < D$ have proved that the evolution can ultimately lead either to the normal homeostasis $(0, 0, d)$ or to the leukemic homeostasis $(0, D, 0)$. One state or the other is reached depending on cell-cell interactions (anti-host, anti-leukemia and anti-graft effects) and initial cell concentrations at transplantation.

The aim of this paper is to find the equilibria of the system (2), to study their stability and to find the boundary of the attraction basins of the stable equilibria. This boundary allows us to calculate an initial concentration (x_0, y_0, z_0) in the basin of attraction of the normal homeostasis $(0, 0, d)$. The next section contains an analysis of the proposed system concerning its dynamics, the third section shows some numerical simulations with physiological parameters and a brief summary is given in the last section.

2. EQUILIBRIA OF THE PROPOSED SYSTEM

Let us consider the system (2) where $a, b, c, A, B, C, g, G, h$ are positive parameters and $x \geq 0, y \geq 0, z \geq 0$ such that $x + y + z > 0$. The main assumptions on the parameters are

$$a > c, \quad A > C, \quad d < D.$$

a) On the Ox axis, we have the equilibrium $P_1(d, 0, 0)$. The eigenvalues of the Jacobian (calculated by MAPLE) at this point are

$$-c < 0, \quad -\frac{c(a-c)}{a} < 0, \quad -\frac{CBa - CBc - Abc + Ccb}{bc + aB - cB} = \frac{BC(D-d)}{1+Bd} > 0$$

so that P_1 is an unstable equilibrium for all considered values of the parameters.

b) On the Oy axis we have the equilibrium $P_2(0, D, 0)$. The eigenvalues of the Jacobian at this point are

$$-c < 0, \quad -\frac{C(A-C)}{A} < 0, \quad \frac{CBa - CBc - Acb + Ccb}{CB + Ab - Cb} = \frac{bc(d-D)}{1+bD} < 0$$

so that P_2 is a stable node for all considered values of the parameters – this is the "bad" equilibrium.

c) On the Oz axis the equilibrium is $P_3(0, 0, d)$. The eigenvalues of the Jacobian at this point are $-c < 0$, $-C < 0$, $-c(a-c)/a < 0$, so that P_3 is a stable node for all considered values of the parameters – this is the "good" equilibrium.

d) In the Oxy plane the equilibrium conditions lead us to the system

$$\left(\frac{a}{1+b(x+y)} - c \right) x = 0, \quad \left(\frac{A}{1+B(x+y)} - C \right) y = 0,$$

from where, for $x > 0$, $y > 0$ we obtain $x + y = d = D$. Consequently, this system is inconsistent for $d < D$ and here we have no equilibria.

e) In the Oxz plane the equilibrium condition leads us to the system

$$(3) \quad \begin{cases} \frac{a}{1+b(x+z)} \frac{x}{x+gz} - c = 0 \\ \frac{a}{1+b(x+z)} \frac{z}{z+hx} - c = 0 \end{cases}$$

from where, for $x > 0$, $z > 0$, we obtain $\frac{x}{x+gz} = \frac{z}{z+hx}$ or $hx^2 = gz^2$.

Consequently, we have a solution $P_4(x^+, 0, z^+)$, where

$$(4) \quad x^+ = \frac{\frac{a}{c(1+\sqrt{hg})} - 1}{b\left(1 + \sqrt{\frac{h}{g}}\right)}, \quad z^+ = \sqrt{h/g}x^+.$$

There exists an admissible solution ($x^+ > 0, z^+ > 0$) if and only if

$$\sqrt{hg} < \frac{a}{c} - 1 = bd.$$

We remark here that $z = \sqrt{h/gx}$ is an invariant manifold in Oxz .

In order to study the stability of this solution we calculate the Jacobian J of the system (2) at this point,

$$J = \begin{pmatrix} -\frac{bcx^+}{1+b(x^++z^+)} + \frac{cgz^+}{x^++gz^+} & \square & -\frac{bcx^+}{1+b(x^++z^+)} - \frac{cgx^+}{x^++gz^+} \\ 0 & Q & 0 \\ -\frac{bcz^+}{1+b(x^++z^+)} - \frac{hcz^+}{z^++hx^+} & \square & -\frac{bcz^+}{1+b(x^++z^+)} + \frac{hcx^+}{z^++hx^+} \end{pmatrix}.$$

An eigenvalue is

$$Q = \frac{Ax^+}{(1+B(x^++z^+))(x^++Gz^+)} - C$$

and from the expressions of x^+ and z^+ we obtain

$$Q = \frac{C}{1 + \frac{B}{b} \left(\frac{a}{c(1+\sqrt{hg})} - 1 \right)} \frac{\frac{A}{C}\sqrt{gh}}{\sqrt{gh} + Gh} - C.$$

The other two eigenvalues have the product

$$\begin{vmatrix} J_{11} & J_{13} \\ J_{31} & J_{33} \end{vmatrix} < 0$$

so that they have opposite signs. Consequently, the equilibrium $(x^+, 0, z^+)$ is unstable. If $Q < 0$ the system (2) has a two-dimensional stable invariant manifold and a one-dimensional unstable invariant manifold. If $Q > 0$ the system has a two-dimensional unstable invariant manifold and a one-dimensional stable invariant manifold, see [4].

f) In the Oyz plane the equilibrium condition leads us to the system

$$(5) \quad \begin{cases} \frac{A}{1+B(y+z)} \frac{y}{y+Gz} = C, \\ \frac{a}{1+b(y+z)} \frac{z}{z+hy} = c, \end{cases}$$

which becomes

$$F_1(y, z) \equiv CBy^2 + CB(G+1)yz + CBGz^2 + (C-A)y + CGz = 0$$

$$F_2(y, z) \equiv cbhy^2 + cb(h+1)yz + cbz^2 + chy + (c-a)z = 0.$$

This means that the equilibrium point is the intersection of these two conics and it is admissible if its coordinates are both positive.

Obviously, the two conics pass through the origin. We also have

$$\delta_1 = \left| \frac{CB}{\frac{CB(G+1)}{2}} \quad \frac{\frac{CB(G+1)}{2}}{CBG} \right| = C^2 B^2 G - \frac{C^2 B^2 (G+1)^2}{4} = -\frac{C^2 B^2 (G-1)^2}{4} < 0$$

for F_1 and

$$\delta_2 = \left| \frac{cbh}{\frac{cb(h+1)}{2}} \quad \frac{\frac{cb(h+1)}{2}}{cb} \right| = c^2 b^2 h - \frac{c^2 b^2 (h+1)^2}{4} = -\frac{c^2 b^2 (h-1)^2}{4} < 0$$

for F_2 so that the conics are hyperbolas if $G \neq 1$, $h \neq 1$.

The calculation of the centers and asymptotes gives us the relative position of these hyperbolas, which have unique branches into the first quadrant of Oyz , see figure 1. The intersection of these branches depends on their slopes at the

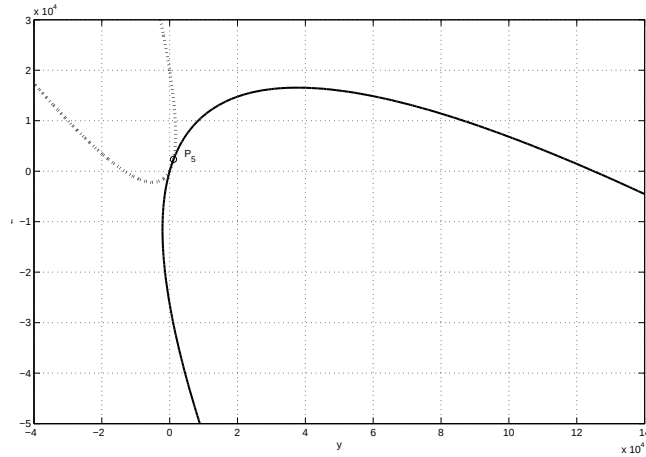


FIGURE 1. The equilibrium point $P_5 \in Oyz$

origin

$$F_1 : z_y = -\frac{F_{1y}(0,0)}{F_{1z}(0,0)} = \frac{A-C}{CG} > 0$$

$$F_2 : z_y = -\frac{F_{2y}(0,0)}{F_{2z}(0,0)} = \frac{ch}{a-c} > 0$$

so that we have an intersection point $P_5(0, y^*, z^*)$ solution of the system (5) with positive coordinates if and only if

$$\frac{A - C}{CG} > \frac{ch}{a - c}$$

i.e.

$$hG < BbDd = \left(\frac{A}{C} - 1 \right) \left(\frac{a}{c} - 1 \right).$$

The stability of this point is given by the eigenvalues of the Jacobian

$$J = \begin{pmatrix} P & 0 & 0 \\ \square & -\frac{BCy^*}{1 + B(y^* + z^*)} + \frac{GCz^*}{y^* + Gz^*} & -\frac{BCy^*}{1 + B(y^* + z^*)} - \frac{GCy^*}{y^* + Gz^*} \\ \square & -\frac{bcz^*}{1 + b(y^* + z^*)} - \frac{chy^*}{z^* + hy^*} & -\frac{bcz^*}{1 + b(y^* + z^*)} + \frac{chy^*}{z^* + hy^*} \end{pmatrix}.$$

One eigenvalue is

$$P = \frac{ay^*}{(1 + b(y^* + z^*))(y^* + gz^*)} - c.$$

The product of the other two eigenvalues is

$$\begin{vmatrix} J_{22} & J_{23} \\ J_{32} & J_{33} \end{vmatrix} < 0$$

so that they have opposite signs. Consequently, the equilibrium $(0, y^*, z^*)$ is unstable. If $P < 0$ the system (2) has a two-dimensional stable invariant manifold and a one-dimensional unstable invariant manifold. If $P > 0$ the system has a two-dimensional unstable invariant manifold and a one-dimensional stable invariant manifold.

From the system (5) verified by (y^*, z^*) , we have

$$P = \frac{a(hy^{*2} - gz^{*2})}{(1 + b(y^* + z^*))(y^* + gz^*)(z^* + hy^*)}$$

whose sign is given by $hy^{*2} - gz^{*2}$. More precisely,

$$\frac{y^*}{z^*} < \sqrt{\frac{g}{h}} \implies P < 0 \text{ and } \frac{y^*}{z^*} > \sqrt{\frac{g}{h}} \implies P > 0$$

In order to evaluate the sign of P , we eliminate $y^* + z^*$ from the system (5) and by denoting $\frac{y^*}{z^*} = t^*$ we have

$$\frac{A}{B} \left(\frac{1}{C} \frac{t^*}{t^* + G} - \frac{1}{A} \right) = \frac{a}{b} \left(\frac{1}{c} \frac{1}{1 + ht^*} - \frac{1}{a} \right).$$

Consequently, t^* is a positive root of the equation

$$f(t) \equiv \frac{A}{BC} \frac{t}{t+G} - \frac{a}{bc} \frac{1}{1+ht} - \frac{1}{B} + \frac{1}{b} = 0.$$

so that

$$\begin{aligned} f\left(\sqrt{\frac{g}{h}}\right) > 0 = f(t^*) &\iff t^* < \sqrt{\frac{g}{h}} \iff P < 0, \\ f\left(\sqrt{\frac{g}{h}}\right) < 0 = f(t^*) &\iff t^* > \sqrt{\frac{g}{h}} \iff P > 0. \end{aligned}$$

Finally, the sign of P is opposite to the sign of

$$f\left(\sqrt{\frac{g}{h}}\right) = \frac{A}{BC} \frac{\sqrt{gh}}{\sqrt{gh}+Gh} - \frac{a}{bc} \frac{1}{1+\sqrt{gh}} - \frac{1}{B} + \frac{1}{b}.$$

If the equilibrium P_5 exists but P_4 does not exist, i.e.

$$Gh < \left(\frac{A}{C} - 1\right) \left(\frac{a}{c} - 1\right), \quad \sqrt{gh} > \frac{a}{c} - 1,$$

then

$$0 < Gh < \left(\frac{A}{C} - 1\right) \sqrt{gh}$$

and

$$\begin{aligned} f\left(\sqrt{\frac{g}{h}}\right) &> \frac{A}{BC} \frac{1}{1 + \frac{(\frac{A}{C}-1)(\frac{a}{c}-1)}{\sqrt{gh}}} - \frac{a}{bc} \frac{1}{1 + \sqrt{hg}} - \frac{1}{B} + \frac{1}{b} > \\ &> \frac{A}{BC} \frac{1}{1 + \frac{A}{C} - 1} - \frac{a}{bc} \frac{1}{1 + \frac{a}{c} - 1} - \frac{1}{B} + \frac{1}{b} = 0. \end{aligned}$$

In this case, $P < 0$ and the equilibrium $P_5(0, y^*, z^*)$ has a two-dimensional stable manifold and a one-dimensional unstable manifold. The unstable manifold is a heteroclinic trajectory to the attractors $P_3(0, 0, d)$ and $P_2(0, D, 0)$ while the stable manifold is a separation boundary between the two attractors P_3 and P_2 .

If the equilibrium P_5 does not exist but P_4 exists, i.e.

$$Gh > \left(\frac{A}{C} - 1\right) \left(\frac{a}{c} - 1\right), \quad \sqrt{gh} < \frac{a}{c} - 1,$$

then

$$Gh > \left(\frac{A}{C} - 1\right) \sqrt{gh}$$

and

$$Q = \frac{C}{1 + \frac{B}{b} \left(\frac{a}{c(1 + \sqrt{hg})} - 1 \right)} \frac{\frac{A}{C} \sqrt{gh}}{\sqrt{gh} + Gh} - C < C \left(\frac{\frac{A}{C} \sqrt{gh}}{\sqrt{gh} + Gh} - 1 \right) < 0.$$

In this case, $Q < 0$ and the equilibrium $P_4(x^+, 0, z^+)$ has a two-dimensional stable manifold and a one-dimensional unstable manifold. The unstable manifold is a heteroclinic trajectory to the equilibria $P_3(0, 0, d)$ and $P_1(d, 0, 0)$ while the stable manifold is a separation boundary between the attraction basins of the two attractors P_3 and P_2 .

If there exist both equilibria P_4 and P_5 , i.e.

$$Gh < \left(\frac{A}{C} - 1 \right) \left(\frac{a}{c} - 1 \right), \quad \sqrt{gh} < \frac{a}{c} - 1,$$

we have

$$Q = \frac{C}{1 + \frac{B}{b} \left(\frac{a}{c(1 + \sqrt{hg})} - 1 \right)} \frac{\frac{A}{C} \sqrt{gh}}{\sqrt{gh} + Gh} - C,$$

$$f \left(\sqrt{\frac{g}{h}} \right) = \frac{A}{BC} \frac{\sqrt{hg}}{\sqrt{hg} + Gh} - \frac{a}{bc} \frac{1}{1 + \sqrt{hg}} - \frac{1}{B} + \frac{1}{b}.$$

By eliminating $\frac{\frac{A}{C} \sqrt{gh}}{\sqrt{gh} + Gh}$ we obtain

$$\frac{\frac{A}{C} \sqrt{gh}}{\sqrt{gh} + Gh} = Bf \left(\sqrt{\frac{g}{h}} \right) + \frac{B}{b} \left(\frac{a}{c} \frac{1}{1 + \sqrt{hg}} - 1 \right) + 1,$$

$$Q = \frac{C \left(Bf \left(\sqrt{\frac{g}{h}} \right) + \frac{B}{b} \left(\frac{a}{c} \frac{1}{1 + \sqrt{hg}} - 1 \right) + 1 \right)}{1 + \frac{B}{b} \left(\frac{a}{c(1 + \sqrt{hg})} - 1 \right)} - C = \frac{BCf \left(\sqrt{\frac{g}{h}} \right)}{1 + \frac{B}{b} \left(\frac{a}{c(1 + \sqrt{hg})} - 1 \right)}$$

Consequently, Q has the same sign as $f \left(\sqrt{\frac{g}{h}} \right)$ so that Q and P have opposite signs. This means that either $P_5(0, y^*, z^*)$ or $P_4(x^+, 0, z^+)$ have a two-dimensional stable manifold which separates the attraction basins of the two attractors P_3 and P_2 .

g) Finally, if we search equilibrium points (x, y, z) with $x > 0$, $y > 0$, $z > 0$, we are led to the system

$$\begin{cases} \frac{a}{1+b(x+y+z)} \frac{x+y}{x+y+gz} - c = 0 \\ \frac{A}{1+B(x+y+z)} \frac{x+y}{x+y+Gz} - C = 0 \\ \frac{a}{1+b(x+y+z)} \frac{z}{z+h(x+y)} - c = 0. \end{cases}$$

In fact, we have 3 equations with 2 unknowns $u = x + y$ and z . This system has solutions only in the very particular case

$$(6) \quad \frac{\frac{A}{C}\sqrt{gh}}{\sqrt{gh} + Gh} = 1 + \frac{B}{b} \left(\frac{\frac{a}{c}}{1 + \sqrt{gh}} - 1 \right)$$

which will be not discussed here.

If we put together the above results, we have the following

Theorem 1. *Let $a, b, c, A, B, C, g, G, h$ be positive parameters, such that $a > c$, $A > C$, $d < D$, where $d = \frac{1}{b} \left(\frac{a}{c} - 1 \right)$ and $D = \frac{1}{B} \left(\frac{A}{C} - 1 \right)$. The system (2), considered for $x \geq 0, y \geq 0, z \geq 0$ has the following steady-states:*

- a) $O(0, 0, 0)$ and $P_1(d, 0, 0)$ as unstable equilibria,
- b) $P_2(0, D, 0)$ and $P_3(0, 0, d)$ as asymptotically stable equilibria,
- c) $P_4(x^+, 0, z^+)$ given by (4) if $hg < \left(\frac{a}{c} - 1 \right)^2$ and
- d) $P_5(0, y^*, z^*)$ given by (5) if $hG < \left(\frac{a}{c} - 1 \right) \left(\frac{A}{C} - 1 \right)$

as unstable equilibria. Only one of P_4 or P_5 has a two-dimensional stable invariant manifold which is also the border between the attraction basins of P_2 and P_3 .

3. NUMERICAL SIMULATIONS

In order to verify and to illustrate the above theoretical results, we will perform some numerical tests, with the values of the parameters chosen from [3]:

$$\begin{aligned} a &= 0.005, & A &= 0.0115, & g &= 2, \\ b &= 0.000075, & B &= 0.000038, & G &= 2, \\ c &= 0.002, & C &= 0.002, & h &= 2, \end{aligned}$$

such that

$$a > c, \quad A > C, \quad d = 20000 < D = 125000$$

and

$$Gh = 4 < \left(\frac{A}{C} - 1\right) \left(\frac{a}{c} - 1\right) = 4.75 \times 1.5$$

$$hg = 4 > \left(\frac{a}{c} - 1\right)^2 = (1.5)^2.$$

Consequently, we have no equilibrium P_4 in the Oxz plane – in this case $x^+ < 0$.

We have $P_5(0, 1149.089506, 2342.140461)$ and the corresponding eigenvalues of the Jacobian are $-.0003471757542$, $.002588209809$, $-.001219452669$. This point P_5 is indeed an unstable equilibrium with a two-dimensional stable manifold which separates the attraction basins of the asymptotically stable equilibria $P_2(0, 125000, 0)$ and $P_3(0, 0, 20000)$, see figure 2.

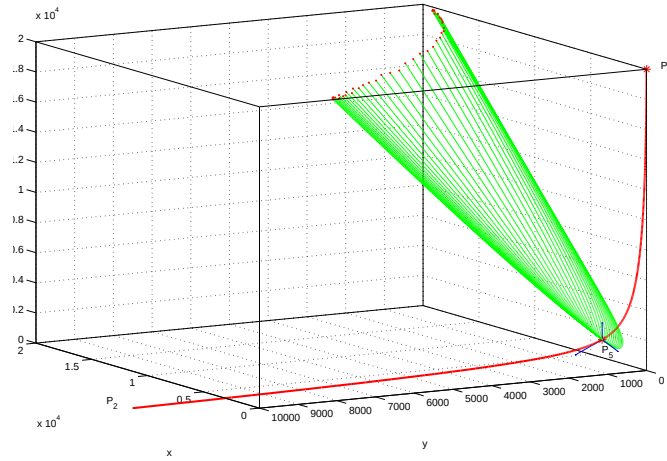


FIGURE 2. The border between the attraction basins of P_2 and P_3 , which is also the two-dimensional stable manifold of P_5 , case $C = 0.002$

There are efficient methods to calculate this two-dimensional stable manifold, see for example [5].

4. CONCLUSION

A mathematical model of Dingli and Michor [3] was augmented in order to simulate the cellular dynamics after bone marrow transplantation. The

new model (2) introduces the parameters h, g, G standing for the intensity of anti-graft, anti-host and anti-leukemia effects.

This model has only two asymptotically stable equilibria, $P_2(0, D, 0)$ and $P_3(0, 0, d)$. Other important unstable hyperbolic equilibria are $P_4(x^+, 0, z^+)$, which exists if and only if $hg < (\frac{a}{c} - 1)^2$, and $P_5(0, y^*, z^*)$, which exists if and only if $hG < (\frac{A}{C} - 1)(\frac{a}{c} - 1)$.

If, after an appropriate therapy, h is sufficiently small, i.e. the anti-graft effect becomes small enough, then one or both equilibria P_4 or P_5 exist and, by the stable manifold theorem for a hyperbolic fixed point, one of them always has a stable two-dimensional invariant manifold.

This stable manifold provides important information about the system's global dynamics, for example it indicates the boundary of the basins of attraction of the asymptotically stable equilibria P_2 and P_3 . Given the initial concentrations x_0, y_0 , we may calculate an initial concentration z_0 in the basin of attraction of P_3 so that, in time, host cells will be eliminated and completely replaced by donor cells, guaranteeing the elimination of cancer.

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DEPARTMENT OF MATHEMATICS, BABEȘ-BOLYAI UNIVERSITY OF CLUJ-NAPOCA, ROMANIA

E-mail address: {r.precup,mserban,dtrif}@math.ubbcluj.ro

ON STARLIKENESS OF BESSEL FUNCTIONS OF THE FIRST KIND

RÓBERT SZÁSZ

ABSTRACT. In this paper a conjecture concerning the starlikeness of Bessel functions is solved. The infinite series and product representations of the Bessel functions of the first kind are the main tool in our proofs.

1. INTRODUCTION

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk. Let $\mathcal{A}$ denote the class of analytic functions defined in the unit disk $\mathbb{D}$, which satisfy $f(0) = f'(0) - 1 = 0$. The subclass $S^* \subset \mathcal{A}$ consists of functions, mapping the unit disc $\mathbb{D}$ on a starlike domain. It is well-known that

$$S^* = \left\{ f \in \mathcal{A} : \operatorname{Re} \frac{zf'(z)}{f(z)} > 0, \ z \in \mathbb{D} \right\}.$$

For $\alpha \in (0, 1)$ let $S^*(\alpha)$ denote the class of starlike functions of order α defined by

$$S^*(\alpha) = \left\{ f \in \mathcal{A} : \operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha, \ z \in \mathbb{D} \right\}.$$

Let $\mathcal{J}_p$ be the function defined by

$$\mathcal{J}_p(z) = 2^p \Gamma(p+1) z^{-p} J_p(z),$$

where J_p is the Bessel function of the first kind

$$J_p(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(n+p+1)} (z/2)^{2n+p}.$$

We define the function $f_p : \mathbb{D} \rightarrow \mathbb{C}$ by

$$f_p(z) = z \mathcal{J}_p(\sqrt{z}) = z + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!(p+1)(p+2) \dots (p+n) 2^{2n}} z^{n+1}.$$

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By using differential subordinations, Selinger investigated in [5] the geometric properties of the mapping $z \mapsto z^{-\frac{p}{2}} J_p(z^{\frac{1}{2}})$. The results of Selinger have been improved in [6] and generalized in [2]. In a recent work [4], which was motivated by [6], the authors studied the starlikeness of the function f_p , and they posed the following conjecture: the function f_p is starlike and starlike of order $\frac{1}{2}$ in $\mathbb{D}$ if and only if $p \geq -0.875$.

We mention that other results concerning this question can be found in [1] and [3]. The aim of this paper is to disprove the above mentioned conjecture and to find the exact range for the order $p > -1$ such that the normalized Bessel function f_p to be starlike of order α . More precisely, we prove that f_p is starlike of order $\alpha \in [0, \alpha_0)$, where $\alpha_0 = f'_1(1)/f_1(1) = 0.8694428678\dots$, if and only if $p \geq p_\alpha$, where p_α is the root of the equation $f'_1(1) = \alpha f_1(1)$ on $(-1, 1)$. For more details see Theorem 6 and Theorem 7 in section 3.

2. PRELIMINARIES

In order to achieve our goal, we first need the following preliminary results.

Lemma 1. *If $z \in \mathbb{C}$, $\alpha \in \mathbb{R}$ and $\alpha > |z|$, then*

$$(1) \quad \frac{|z|}{\alpha - |z|} \geq \operatorname{Re} \left(\frac{z}{\alpha - z} \right).$$

Proof. Let $z = x + iy$ and $m = |z| = \sqrt{x^2 + y^2}$. After replacing, we obtain that the inequality is equivalent to

$$\frac{m}{\alpha - m} \geq \frac{\alpha x - m^2}{\alpha^2 - 2\alpha x + m^2}.$$

and this is equivalent to $(m - x)(\alpha + m)\alpha \geq 0$. This means that the inequality (1) holds. Equality occurs if and only if z is real and positive.

We will need in our study the following result of von Lommel and the infinite product representation given in [4, p. 498].

Lemma 2. *If $p > -1$ then all the zeros of the Bessel function J_p are real. Denoting the n th positive root of J_p by $j_{p,n}$, the following Weierstrassian factorization is valid*

$$(2) \quad J_p(z) = \frac{z^p}{2^p \Gamma(p+1)} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{j_{p,n}^2} \right),$$

where the infinite product is uniformly convergent on each compact subset of $\mathbb{C}$.

Lemma 3. *The equation $f_p(1) = 0$ has a single root $p^* \in (-1, 1)$, $p^* \simeq -0.7745\dots$. If $p \geq p^*$, then the smallest positive root of f_p is $j_{p,1}^2$ and $j_{p,1} > 1$. Moreover, if $p \in (-1, p^*)$, then $j_{p,1} \in (0, 1)$.*

Proof. Let $v : (-1, \infty) \rightarrow \mathbb{R}$ be the function defined by:

$$v(p) = f_p(1) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n \Gamma(p+1)}{n! \Gamma(p+n+1) 2^{2n}}.$$

Since

$$\begin{aligned} v'(p) &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \Gamma(p+1)}{n! \Gamma(p+n+1) 2^{2n}} \left(\sum_{k=1}^n \frac{1}{p+k} \right) \\ &= \sum_{n=1}^{\infty} \frac{\Gamma(p+1)}{(2n-1)! \Gamma(p+2n) 2^{4n-2}} \left(\sum_{k=1}^{2n-1} \frac{1}{p+k} - \frac{1}{8n(p+2n)} \sum_{k=1}^{2n} \frac{1}{p+k} \right) > 0 \end{aligned}$$

for all $p > -1$, the function v is strictly increasing. Now because $v(1) = f_1(1) = \mathcal{J}_1(1) = 2J_1(1) \simeq 0.880101171\dots > 0$, and $\lim_{p \searrow -1} v(p) = -\infty$, this implies that the equation $v(p) = 0$ has exactly one root $p^* \in (-1, 1)$, and $v(p) < 0$ for $p \in (-1, p^*)$ and $v(p) > 0$ for $p \in (p^*, \infty)$.

On the other hand, von Lommel's celebrated result says that if $p > -1$, then the zeros of the Bessel function J_p are real, or equivalently, the equation $f_p(x) = 0$ has only real roots. A short calculation leads to

$$\begin{aligned} f_p''(x) &= \sum_{n=1}^{\infty} \frac{(-1)^n n(n+1)}{n! (p+1)(p+2)\dots(p+n) 2^{2n}} x^{n-1} \\ &= - \sum_{n=1}^{\infty} \frac{2n(2n-1) \Gamma(p+1)}{(2n-1)! \Gamma(p+2n) 2^{4n-2}} x^{2n-2} \left(1 - \frac{2n+1}{4(2n-1)2n(p+2n)} x \right). \end{aligned}$$

This shows that the inequality $f_p''(x) < 0$ holds for every $x \in (0, 1)$, and $p > -1$ consequently, it follows that f_p is concave on $(0, 1)$ for all $p > -1$. Thus, in case of $p \in [p^*, 1)$, the equalities $f_p(0) = 0$ and $f_p(1) = v(p) \geq 0$ imply that the equation $f_p(x) = 0$ has no roots in $(0, 1)$. This means that, the smallest positive root of $f_p(x) = 0$ is not smaller than one.

In case of $p \in (-1, p^*)$, we have $f_p'(0) = 1$. The continuity of f_p' in $x = 0$ implies that there exists a positive number $\epsilon \in (0, 1)$ such that $f_p'(x) > 0$, $x \in (0, \epsilon)$. Thus $f_p(0) = 0$ leads to $f_p(\epsilon) > 0$. Since $f_p(1) = v(p) < 0$, it follows that the equation $f_p(x) = 0$ has a root in $(0, 1)$.

Lemma 4. *If $p \in [p^*, p_0)$, where $p_0 = -0.5623\dots$ is the root of the equation $f_p'(1) = 0$, then the equation $f_p'(x) = 0$ has a real root that lies in $(0, 1)$.*

Proof. Since $f_p'(0) = 1$, and in view of the proof of Lemma 2 the function f_p' is strictly decreasing on $(0, 1)$, it is enough to show that $f_p'(1) < 0$, for every $p \in (p^*, p_0)$. In order to do this, we study the function $w : (-1, \infty) \rightarrow \mathbb{R}$

defined by

$$w(p) = f'_p(1) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n(n+1)}{n!(p+1)(p+2)\dots(p+n)2^{2n}}.$$

It can be shown that for all $p > -1$ we have

$$\begin{aligned} w'(p) &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n+1)\Gamma(p+1)}{n!\Gamma(p+n+1)2^{2n}} \left(\sum_{k=1}^n \frac{1}{p+k} \right) \\ &= \sum_{n=1}^{\infty} \frac{2n\Gamma(p+1)}{(2n-1)!\Gamma(p+2n)2^{4n-2}} \left[\sum_{k=1}^{2n-1} \frac{1}{p+k} \right. \\ &\quad \left. - \frac{2n+1}{16n^2(p+2n)} \left(\sum_{k=1}^{2n} \frac{1}{p+k} \right) \right] > 0. \end{aligned}$$

Thus, the function w is strictly increasing. On the other hand,

$$\begin{aligned} w(p^*) &= w(p^*) - v(p^*) = \sum_{n=1}^{\infty} \frac{(-1)^n}{(n-1)!(p^*+1)\dots(p^*+n)2^{2n}} \\ &< -\frac{1}{4(p^*+1)} \left(1 - \sum_{n=1}^{\infty} \frac{1}{2^{2n}} \right) = -\frac{1}{6(p^*+1)} < 0 \end{aligned}$$

and

$$w(1) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!2^{2n}} > 1 - \sum_{n=1}^{\infty} \frac{1}{2^{2n}} = \frac{2}{3} > 0.$$

These imply that the equation $w(p) = 0$ has a single root $p_0 \in (p^*, 1)$, and also that $w(p) < w(p_0) = 0$ for every $p \in (p^*, p_0)$. Consequently, if $p \in [p^*, p_0)$, the equation

$$f'_p(x) = 0$$

has a root in $(0, 1)$ because $f'_p(0) = 1 > 0$ and $f'_p(1) = w(p) < 0$.

Lemma 5. *If $\alpha \in [0, \alpha_0)$, where $\alpha_0 = f'_1(1)/f_1(1) \simeq 0.8694428678\dots$, then for $p \in (-1, 1)$, the equation $f'_p(1) = \alpha f_p(1)$ has a unique simple solution.*

Proof. Let us consider the function $u : (-1, \infty) \rightarrow \mathbb{R}$ defined by

$$u(p) = f'_p(1) - \alpha f_p(1) = 1 - \alpha + \sum_{n=1}^{\infty} \frac{(-1)^n(n+1-\alpha)}{n!(p+1)(p+2)\dots(p+n)2^{2n}}.$$

A similar calculation to a previous one, used in the proof of Lemma 3, leads to

$$\begin{aligned} u'(p) &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n+1-\alpha)\Gamma(p+1)}{2^{2n}n!\Gamma(n+p+1)} \left(\sum_{k=1}^n \frac{1}{p+k} \right) \\ &= \sum_{n=1}^{\infty} \frac{(2n-\alpha)\Gamma(p+1)}{(2n-1)!\Gamma(p+2n)2^{4n-2}} \left[\sum_{k=1}^{2n-1} \frac{1}{p+k} \right. \\ &\quad \left. - \frac{2n+1-\alpha}{8n(p+2n)(2n-\alpha)} \left(\sum_{k=1}^{2n} \frac{1}{p+k} \right) \right]. \end{aligned}$$

Observe that for $\alpha \in (0, 1)$, $n \in \{1, 2, 3, \dots\}$ and $p \in (-1, 1)$, the inequalities

$$\frac{2n+1}{2n} \leq \frac{2n+1-\alpha}{2n-\alpha} \leq \frac{2n}{2n-1},$$

and

$$0 < \frac{1}{p+2n} < \frac{1}{2n-1},$$

imply

$$(3) \quad 0 < \frac{2n+1-\alpha}{8n(p+2n)(2n-\alpha)} \leq \frac{1}{4(2n-1)^2} \leq \frac{1}{4}.$$

Now from (3) for all $p \in (-1, 1)$ we get

$$\begin{aligned} u'(p) &> \sum_{n=1}^{\infty} \frac{(2n-\alpha)\Gamma(p+1)}{(2n-1)!\Gamma(p+2n)2^{4n-2}} \left[\sum_{k=1}^{2n-1} \frac{1}{p+k} - \frac{1}{4} \sum_{k=1}^{2n} \frac{1}{p+k} \right] \\ &= \sum_{n=1}^{\infty} \frac{(2n-\alpha)\Gamma(p+1)}{(2n-1)!\Gamma(p+2n)2^{4n-2}} \left[\frac{3}{4} \sum_{k=1}^{2n-1} \frac{1}{p+k} - \frac{1}{4} \frac{1}{p+2n} \right] > 0. \end{aligned}$$

Summarizing, the function u is strictly increasing, and because

$$\lim_{p \searrow -1} u(p) = -\infty, \quad u(1) \geq f'_1(1) - \alpha f_1(1) > f'_1(1) - \alpha_0 f_1(1) = 0$$

the proof is complete.

3. STARLIKENESS OF BESSEL FUNCTIONS OF THE FIRST KIND

The first main result of this paper reads as follows.

Theorem 6. *The function f_p is starlike if and only if $p \geq p_0$.*

Proof. Lemma 3 implies that if $p \in (-1, p^*]$, then the function f_p has a root in $\mathbb{D}$, which is different from zero. Since $f_p(0) = 0$, the function f_p is not univalent in $\mathbb{D}$ for $p \in (-1, p^*)$, and consequently f_p is not a starlike function in $\mathbb{D}$. According to Lemma 4, if $p \in [p^*, p_0)$, then the derivative f'_p has a root in $\mathbb{D}$, consequently the function f_p cannot be univalent in $\mathbb{D}$, so it cannot be starlike either. These in turn imply that if $p < p_0$, then f_p is not starlike in $\mathbb{D}$, or equivalently if f_p is starlike in $\mathbb{D}$, then we necessarily have $p \geq p_0$. In what follows, our aim is to show that if $p \geq p_0$, then f_p is starlike in $\mathbb{D}$. The logarithmic differentiation of the equalities $f_p(z) = z\mathcal{J}_p(\sqrt{z})$, and (2) leads to $\frac{zf'_p(z)}{f_p(z)} = 1 - \frac{p}{2} + \frac{1}{2} \frac{z^{1/2}J'_p(z^{1/2})}{J_p(z^{1/2})}$, and $\frac{zJ'_p(z)}{J_p(z)} = p - \sum_{n=1}^{\infty} \frac{2z^2}{j_{p,n}^2 - z^2}$. Consequently, it follows that

$$\frac{zf'_p(z)}{f_p(z)} = 1 - \sum_{n=1}^{\infty} \frac{z}{j_{p,n}^2 - z}.$$

By using Lemma 1, and the fact that according to Lemma 3 if $p > p^* = -0.774\dots$, then

$$\dots > j_{p,n} > \dots > j_{p,1} > j_{p^*,1} = 1, \quad \text{for all } n \in \{2, 3, 4, \dots\},$$

we obtain for $z \in \mathbb{D}$

$$\operatorname{Re} \frac{zf'_p(z)}{f_p(z)} = 1 - \operatorname{Re} \sum_{n=1}^{\infty} \frac{z}{j_{p,n}^2 - z} \geq 1 - \sum_{n=1}^{\infty} \frac{|z|}{j_{p,n}^2 - |z|} = \frac{|z|f'_p(|z|)}{f_p(|z|)}.$$

Now consider the function $h : (0, j_{p,1}^2) \rightarrow \mathbb{R}$ defined by

$$h(r) = \frac{rf'_p(r)}{f_p(r)} = 1 - \sum_{n=1}^{\infty} \frac{r}{j_{p,n}^2 - r}.$$

Since for all $p > -1$ and $r \in (0, j_{p,1}^2)$ we have

$$h'(r) = - \sum_{n=1}^{\infty} \frac{j_{p,n}^2}{(j_{p,n}^2 - r)^2} < 0$$

it follows that h is strictly decreasing and finally for all $z \in \mathbb{D}$ we get that

$$(4) \quad \operatorname{Re} \frac{zf'_p(z)}{f_p(z)} \geq h(|z|) > h(1) = f'_p(1)/f_p(1).$$

Now recall that according to the proof of Lemma 3 and Lemma 4 the functions v and w are strictly increasing, and consequently for all $p \geq p_0$ we have

$$f'_p(1) = w(p) \geq w(p_0) = 0$$

and

$$f_p(1) = v(p) \geq v(p_0) > v(p^*) = 0.$$

These in turn imply that if $p \geq p_0$, then $f'_p(1)/f_p(1) \geq 0$ and consequently indeed f_p is starlike in $\mathbb{D}$. It is important to note that following the proof of Theorem 2 and by using Lemma 5, we may get the following generalized form of the previous result.

Theorem 7. *The function f_p is starlike of order $\alpha \in [0, \alpha_0)$, where $\alpha_0 = f'_p(1)/f_p(1) = 0.869\dots$, if and only if $p \geq p_\alpha$, where p_α is the unique root of $f'_p(1) - \alpha f_p(1) = 0$, $p \in (-1, 1)$.*

Proof. Now recall that in view of the proof of Lemma 5, the function u is strictly increasing. The inequality $0 \geq -\alpha v(p_0)$ is equivalent to $u(p_\alpha) \geq u(p_0)$ and consequently $p_\alpha \geq p_0$. Furthermore the monotonicity of u implies that for all $p \geq p_\alpha$ we have

$$u(p) = f'_p(1) - \alpha f_p(1) \geq u(p_\alpha) = 0.$$

On the other hand, because $p \geq p_\alpha \geq p_0 > p^*$ and the function v is strictly increasing, we obtain that:

$$f_p(1) = v(p) \geq v(p_\alpha) \geq v(p_0) > v(p^*) = 0.$$

These in turn imply that if $p \geq p_\alpha$ then we have $f'_p(1)/f_p(1) \geq \alpha$, which according to (4) implies that f_p is starlike of order α . Analogously can be proved that $p < p_\alpha$ implies

$$(5) \quad \frac{f'_p(1)}{f_p(1)} < \alpha.$$

Since $\lim_{\substack{z \rightarrow 1 \\ |z| < 1}} \operatorname{Re} \frac{zf'_p(z)}{f_p(z)} = \frac{f'_p(1)}{f_p(1)}$ the inequality (5) implies that there is a point $z_0 \in \mathbb{D}$ such that

$$\operatorname{Re} \frac{z_0 f'_p(z_0)}{f_p(z_0)} < \alpha.$$

This means that if $p < p_\alpha$, then f_p is not starlike of order α .

In particular, Theorem 7 yields the following result which disproves the third conjecture from [4].

Collorary 8. *f_p is starlike of order $1/2$ if and only if $p \geq p_{1/2}$, where $p_{1/2} = -0.34009\dots$ is the unique root of the equation $2f'_p(1) = f_p(1)$ for $p \in (-1, 1)$.*

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DEPARTMENT OF MATHEMATICS AND INFORMATICS, SAPIENTIA UNIVERSITY, TG. MUREŞ,
ROMANIA

E-mail address: rszasz@ms.sapientia.ro

CHARACTERIZATION OF SEMI-CNS POLYNOMIALS

PÉTER VARGA

ABSTRACT. Let $P(x) = c_0 + c_1x + \dots + c_{n-1}x^{n-1} \in \mathbb{Z}[x]$ and $D = \{0, 1, \dots, |c_0| - 1\}$. The polynomial $P(x)$ is called CNS-polynomial if every coset of the factor ring $\mathbb{Z}[x]/P(x)\mathbb{Z}[x]$ has a member of form

$$(1) \quad \sum_{h=0}^{\infty} d_h x^h,$$

with $d_h \in D, h = 0, 1, \dots$ and such that only finitely many d_h are non-zero. This concept generalizes the negative-base radix representation of integers. It was introduced and studied in [2]. The characterization of CNS polynomials already for degree three is complicated, as indicated in [3]. It is still unsolved.

Burcsi and Kovács [1] called $P(x)$ a *semi-CNS polynomial* if the finite expansions (1) form an additive semigroup. This is a generalization of the usual radix representations of natural numbers. They were able to prove some sufficient properties for $P(x)$ being a semi-CNS polynomial. Moreover they generalized Brunotte's algorithm for semi-CNS polynomials.

In this talk, which is based on a joint work with A. Pethő we give a complete characterization of cubic semi-CNS polynomials. More precisely, in all those polynomials, which do not satisfy the condition given by Burcsi and Kovács, are not semi-CNS. To prove this we present a cycle for each polynomials.

1. INTRODUCTION

In this section we will introduce some necessary definitions, and after those we will speak about some properties and an algorithm, which will help us during the proof of our result.

Definition 1. Let $P(x) = c_0 + c_1x + \dots + c_{n-1}x^{n-1} \in \mathbb{Z}[x]$ and $D = \{0, 1, \dots, |c_0| - 1\}$. The polynomial $P(x)$ is called **CNS polynomial** if every coset of

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the factor ring $\mathbb{Z}[x]/P(x)\mathbb{Z}[x]$ has a member of form

$$(2) \quad \sum_{h=0}^{\infty} d_h x^h,$$

with $d_h \in D, h = 0, 1, \dots$ and such that only finitely many d_h are non-zero.

An equivalent definition for CNS polynomial is the following:

If the set of the residue polynomials of the factor ring mentioned above forms an additive group, then we call $P(x)$ CNS polynomial.

The pair (P, D) called **Canonical Number System (CNS)**, where D is called **digit set**.

Definition 2. Let $P(x) = c_0 + c_1x + \dots + c_{n-1}x^{n-1} \in \mathbb{Z}[x]$ and $D = \{0, 1, \dots, |c_0| - 1\}$. The polynomial $P(x)$ is called **Semi-CNS polynomial** if the set of the residue polynomials of the factor ring $\mathbb{Z}[x]/P(x)\mathbb{Z}[x]$ forms an additive semigroup.

Definition 3. Let $P(x)$ be a polynomial with degree n and let the complex roots of P are $\alpha_1, \alpha_2, \dots, \alpha_n$. If $|\alpha_i| > 1$ for all $i \in 1, 2, \dots, n$ then P is called **expansive polynomial**.

Remark 4. CNS and semi-CNS polynomials must be expansive.

Remark 5. The whole characterization of the CNS and semi-CNS polynomials is still an open problem. Although the characterization of CNS polynomials already for degree three is complicated, as indicated in [3], the characterization for semi-CNS polynomials for degree three will be solved in this paper.

Remark 6. In [1], the Theorem 3.4 says:

Let $c_1 + c_1x + \dots + c_{n-1}x^{n-1} + x^n$ be a polynomial with $c_0 < 0$. If no other coefficient is negative, then it is a semi-CNS polynomial.

Definition 7. Let the following algorithm called **"Backward" division process**:

Let $P(x)$ be the polynomial we want to investigate. Let $P(x)$ be a monic polynomial over $\mathbb{Z}$ with degree n . This algorithm starts from an arbitrary polynomial with degree $n - 1$, which will be called witness. In every step of the algorithm this transformation will be used:

$$a_{n-1}x^{n-1} + \dots + a_1x + a_0 \longrightarrow \frac{a_{n-1}x^{n-1} + \dots + a_1x + a_0 - qP(x) - (a_0 - qc_0)}{x},$$

where c_0 is the constant coefficient of $P(x)$ and q is an integer value, which is determined by the expression $a_0 - qc_0 \in [0, -c_0 - 1]$.

We will mark this step like this:

$$(a_0, a_1, \dots, a_{n-1}) \xrightarrow{q} [a_0 - qc_0](b_0, b_1, \dots, b_{n-1}),$$

where the b vector contains the coefficients of the created polynomial in revert order.

If this algorithm runs into a cycle, then we say that the starter polynomial is a witness in order to $P(x)$ is not a semi-CNS polynomial.

This algorithm can be found in the [2]'s Section 3.

The next theorem is the main result of this article. This will give us the characterization of the cubic semi-CNS polynomials.

Theorem 8. *Let $\mathcal{P}$ be the set of monic cubic expansive polynomials with integer coefficients. Let $P \in \mathcal{P}$ be a polynomial with negative constant. If there is another negative coefficient in the polynomial, then it cannot be a semi-CNS polynomial.*

2. THE PROOF OF THEOREM 8

2.1. The triangle of Cubic expansive polynomials. Let $\mathcal{P}$ be the set of monic cubic polynomials with integer coefficients.

$$P(x) := x^3 + c_2x^2 + c_1x - c_0, \quad c_0 \in \mathbb{Z}^-, \quad (c_1, c_2) \in \mathbb{Z} \times \mathbb{Z}, \quad P(x) \in \mathcal{P}.$$

Let $f_p : \mathcal{P} \rightarrow \mathbb{Z} \times \mathbb{Z}$ be a bijective function:

$$f_{c_0}(x^3 + c_2x^2 + c_1x - c_0) = (c_2, c_1),$$

and let $g : \mathcal{P} \rightarrow \{0, 1\}$ be a logical function, which says the polynomial is semi-CNS or not. Let's see these in a lattice with a parameter c_0 . The function f_{c_0} says the coordinates of a polynomial, the function g says the value of the point with these coordinates.

Here we can use Remark 2 for the polynomials in $\mathcal{P}$, so if $P(x) \in \mathcal{P}$ and $P(x)$ has a root with absolute value less than 1, then $g(P(x)) = 0$.

$P(x)$ has 3 roots in the complex field (comes from the Fundamental theorem

of algebra): r_1, r_2, r_3 . Let's see the Viète formula for $P(x)$.

$$c_0 = r_1 r_2 r_3$$

So, if we find a root of $P(x)$, which is on the real axle and its absolute value less then or equal to $|c_0|$, then there will be a root with absolute value less than 1, because in these cases the Viète formula says:

$$|r_1| \geq c_0 \rightarrow r_2 r_3 \leq 1 \rightarrow |r_2| \leq 1 \text{ or } |r_3| \leq 1.$$

For all $P(x) \in \mathcal{P}$:

- $P(0) = -p$, so this value is negative,
- $\lim_{x \rightarrow -\infty} P(x) = -\infty$, so this value is negative as well,
- $\lim_{x \rightarrow +\infty} P(x) = +\infty$, so this value is positive.

If $P(-c_0)$ is positive or zero, there is necessarily a root between $-\infty$ and $-c_0$.

If $P(-1)$ is positive or zero, there is necessarily a root between -1 and 0 .

If $P(1)$ is positive or zero, there is necessarily a root between 0 and 1 .

If $P(c_0)$ is negative or zero, there is necessarily a root between c_0 and ∞ .

The following sentence describes these conditions, so this must be true for a cubic monic semi-CNS polynomial:

$$P(-c_0) < 0 \text{ AND } P(-1) < 0 \text{ AND } P(1) < 0 \text{ AND } P(c_0) > 0$$

With this method we are able to close out all the cubic polynomials which are not expansive, because every polynomial over Z has a root r which is real and the other two roots' product must be real, too.

$$0 > P(-c_0) = -c_0^3 + c_2 c_0^2 - c_1 c_0 - c_0 \rightarrow c_1 > -c_0^2 + c_2 c_0 - 1$$

$$0 > P(-1) = -1 + c_2 - c_1 - c_0 \rightarrow c_1 > -c_0 + c_2 - 1$$

$$0 > P(1) = 1 + c_2 + c_1 - c_0 \rightarrow c_1 < c_0 - c_2 - 1$$

$$0 < P(c_0) = c_0^3 + c_2 c_0^2 + c_1 c_0 - c_0 \rightarrow c_1 > -c_0^2 - c_2 c_0 + 1$$

This will define a triangle on the plain of the domain of f_{c_0} with apices: $(-c_0 + 2, -2c_0 + 2)$, $(c_0 - 1, -1)$, $(-c_0 - 1, 2c_0 - 1)$. With the help of Remark 3 we already know semi-CNS polynomials, who hasn't got any other negative coefficient just the constant one. These are the green points of Figure 1.

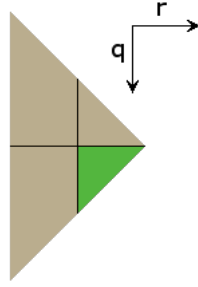


FIGURE 1. Semi-CNS polynomials in the triangle.

If we can investigate the all other points of the triangle, which has another negative coefficients in addition to the constant one, then we can say that we investigated the all cubic monic polynomials. These points are in the brown field in Figure 1.

Remark 9. *The following sections will present a proof for the subparts of the cubic expansive polynomials' triangle. In these sections we define $p, q, r \in \mathbb{N}$ to speak about the coefficients of the investigated polynomials. These parameters are defined to be not negative in order to obtain an understandable proof. If the coefficient of the polynomial is negative then we use the minus signature before the defined parameters.*

2.2. The right-upper-triangle. First, let's examine those polynomials, where the square term's coefficient is positive and the linear term's coefficient is negative (see in Figure 2.):

$$P(x) := x^3 + rx^2 - qx - p,$$

$$1 \leq p < +\infty,$$

$$1 \leq r \leq p - 1,$$

$$1 \leq q \leq p - r.$$

Let's start the "Backward" division process with these conditions from $(-r, -1)$

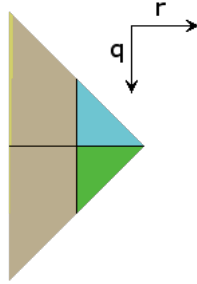


FIGURE 2. Blue: Actually examined points - Green: Semi-CNS polynomials - Yellow: Not semi-CNS polynomials

values:

$$(-r, -1, 0) \xrightarrow{1} [p - r](q - 1, -r, -1)$$

In this line, the $p - r$ tag is the first, which is acceptable, because it is in the $[0, p - 1]$ interval. The next is the $q - 1$. Let's see the upper and the lower bound of this value:

Lower bound of $q - 1$:

$$q - 1 \geq 1 - 1 = 0$$

Upper bound of $q - 1$:

$$q - 1 \leq p - r - 1 \leq p - 2$$

So, this value is also in the interval.

We found a one step cycle with values $(-r, -1)$, we can say that these polynomials are not semi-CNS polynomials.

2.3. The upper-square. In this part of the proof we will prove the upper square of the triangle, which can be seen in Figure 3.

$$P(x) := x^3 - rx^2 - qx - p,$$

$$1 \leq p < +\infty,$$

$$0 \leq r \leq p - 1,$$

$$0 \leq q \leq p - 1.$$

Let's see the algorithm from -1 :

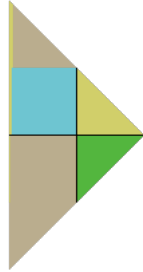


FIGURE 3. Blue: Actually examined points - Green: Semi-CNS polynomials - Yellow: Not semi-CNS polynomials

$$(-1, 0, 0) \xrightarrow{1} [p - 1](q, r, -1)$$

$p - 1, q, r$ are acceptable values, so:

$$(q, r, -1) \xrightarrow{0} [q](r, -1, 0) \xrightarrow{0} [r](-1, 0, 0)$$

here is a three-step cycle with the -1 value.

2.4. The lower-square. In this part of the proof we will prove the lower square of the triangle. The first two subpart is shown in Figure 4 with blue color. After that we prove the lower borderline, and one missed point on the lower-left corner.

2.4.1. *First triangle.*

$$P(x) := x^3 - rx^2 + qx - p,$$

$$1 \leq p < +\infty,$$

$$1 \leq r \leq p,$$

$$1 \leq q \leq r.$$

Now, we start the algorithm from -1 :

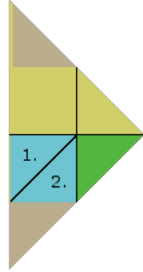


FIGURE 4. Blue: Actually examined points - Green: Semi-CNS polynomials - Yellow: Not semi-CNS polynomials

$$(-1, 0, 0) \xrightarrow{1} [p-1](-q, r, -1)$$

Lower bound of $-q$:

$$-q \geq -r \geq -p.$$

Upper bound of $-q$:

$$-q \leq -1.$$

So, the next step:

$$(-1, 0, 0) \xrightarrow{1} [p-1](-q, r, -1) \xrightarrow{1} [p-q](-q+r, r-1, -1)$$

Lower bound of $-q+r$:

$$-q+r \geq -r+r=0.$$

Upper bound of $-q+r$:

$$-q+r \leq -1+r \leq -1+p.$$

$-q+r, r-1$ is also acceptable values, so:

$$(-q+r, r-1, -1) \xrightarrow{0} [-q+r](r-1, -1, 0) \xrightarrow{0} [r-1](-1, 0, 0)$$

We found a four-steps cycle with -1 .

2.4.2. *Second triangle.*

$$P(x) := x^3 - rx^2 + qx - p,$$

$$1 \leq p < +\infty,$$

$$1 \leq r \leq p,$$

$$r \leq q \leq p.$$

Now, we start the algorithm from $(-q+r-1, r-1, -1)$ values. Lower bound of $-q+r-1$:

$$-q+r-1 \geq -p+r-1 \geq -p.$$

Upper bound of $-q+r-1$:

$$-q+r-1 \leq -r+r-1 = -1.$$

$$(-q+r-1, r-1, -1) \xrightarrow{1} [p-q+r-1](-q+r-1, r-1, -1)$$

So we found a single cycle with values: $-q + r - 1, r - 1, -1$.

2.4.3. *Lower borderline.*

$$P(x) := x^3 - rx^2 + qx - p,$$

$$1 \leq p < +\infty,$$

$$q = p + 1,$$

$$2 \leq r \leq p + 1.$$

The upper bound of r is acceptable because of the big triangle's bounds. Let's investigate it with the "Backward" division process started from $(-q + r - 1, r - 1, -1)$.

Lower bound of $-q + r - 1$:

$$-q + r - 1 = -p + r - 2 \geq -p + 2 - 2 = -p.$$

Upper bound of $-q + r - 1$:

$$-q + r - 1 = -p + r - 2 \leq -p + p + 1 - 2 = -1.$$

$$(-q + r - 1, r - 1, -1) \xrightarrow{1} [p - q + r - 1](-q + r - 1, r - 1, -1)$$

So, this is a single cycle with values: $-q + r - 1, r - 1$ and -1 .

2.4.4. *Lower-left corner.* Here we present the last point which is still out of the investigation.

$$P(x) := x^3 - rx^2 + qx - p,$$

$$1 \leq p < +\infty,$$

$$r = p + 1,$$

$$q = p.$$

$$\begin{aligned} (p, 0, 0) &\xrightarrow{-1} [0](p, -p - 1, 1) \xrightarrow{-1} [0](-1, -p, 1) \xrightarrow{1} [p - 1](-2p, p + 2, -1) \xrightarrow{2} \\ &\xrightarrow{2} [0](-p + 2, 2p + 1, -2) \xrightarrow{1} [2](p + 1, p - 1, -1) \xrightarrow{-1} [1](2p - 1, -p - 2, 1) \xrightarrow{-1} \\ &\xrightarrow{-1} [p - 1](-2, -p, 1) \xrightarrow{1} [p - 2](-2p, p + 2, -1) \end{aligned}$$

So, here is a five steps cycle with values: $-2p$, $p + 2$ and -1 .

2.5. The upper-triangle. In the part, we will investigate the upper part of the whole triangle.

$$\begin{aligned} P(x) &:= x^3 - rx^2 - qx - p, \\ 3 &\leq p < +\infty, \\ 0 &\leq r \leq p - 2, \\ p &\leq q \leq p + r. \end{aligned}$$

Let the algorithm starts with -1 :

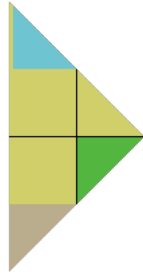


FIGURE 5. Blue: Actually examined points - Green: Semi-CNS polynomials - Yellow: Not semi-CNS polynomials

$$(-1, 0, 0) \xrightarrow{1} [p-1](q, r, -1)$$

Lower bound of q :

$$q \geq p.$$

Upper bound of q :

$$\begin{aligned} q &\leq p + r \leq p + p - 2 = 2p - 2. \\ (-1, 0, 0) &\xrightarrow{1} [p-1](q, r, -1) \xrightarrow{-1} [-p+q](-q+r, -r-1, 1) \end{aligned}$$

Lower bound of $-q + r$:

$$-q + r \geq -(p + r) + r = -p.$$

Upper bound of $-q + r$:

$$-q + r \leq -p + r \leq -p + p - 2 \leq -2.$$

$$\begin{aligned} (-1, 0, 0) &\xrightarrow{1} [p-1](q, r, -1) \xrightarrow{-1} [-p+q](-q+r, -r-1, 1) \xrightarrow{1} \\ &\xrightarrow{-1} [p-q+r](q-r-1, r+1, -1) \end{aligned}$$

Lower bound of $q - r - 1$:

$$q - r - 1 \geq p - r - 1 \geq p - p + 2 - 1 = 1.$$

Upper bound of $q - r - 1$:

$$q - r - 1 \leq p + r - r - 1 = p - 1.$$

Lower bound of $r + 1$:

$$r + 1 \geq 1.$$

Upper bound of $r + 1$:

$$r + 1 \leq p - 2 + 1 = p - 1.$$

So, here is a three steps cycle with value -1 .

2.6. The lower-triangle. In the part, we will investigate the lower part of the whole triangle. There are two subpart of this section, where we investigate two triangle-series as you can see in Figure 7 and Figure 8.

In this section we can say that $p \geq 3$ because in the $p < 3$ case this triangle not exists (if $p = 3$, the triangle is just one point).

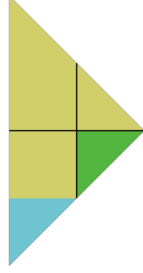


FIGURE 6. Blue: Actually examined points - Green: Semi-CNS polynomials - Yellow: Not semi-CNS polynomials

2.6.1. First triangle-series.

$$P(x) := x^3 - rx^2 + qx - p,$$

$$3 \leq p < +\infty,$$

$$1 \leq s \leq p \text{ (the number of the actual triangle in the series),}$$

$$\frac{ps+1}{s+1} + 2 \leq r \leq p+1,$$

$$p + \frac{s-1}{s}r + \frac{3}{s} - 1 \leq q \leq p + r - 1 + \frac{1-p}{s+1}.$$

Let's start "Backward" division process from the $(-2q + r, 2r - 1, -2)$ values. Now we should investigate $-2q + r$, but instead of it, we generally investigate the $-tq + (t-1)r - (t-2)$ expression, where $2 \leq t \leq s$:
Lower bound of $-tq + (t-1)r - (t-2)$:

$$\begin{aligned} -tq + (t-1)r - (t-2) &\geq -t \left(p + r - 1 + \frac{1-p}{s+1} \right) + (t-1)r - (t-2) = \\ &= -pt + t \frac{p-1}{s+1} - r + 2 \geq -pt + t \frac{p-1}{s+1} - p + 1 > \\ &> -pt - p = (-t-1)p. \end{aligned}$$



FIGURE 7. The lower triangle, first triangle-series. Blue: Actually examined points

Upper bound of $-tq + (t-1)r - (t-2)$:

$$\begin{aligned}
 -tq + (t-1)r - (t-2) &\leq -t \left(p + \frac{s-1}{s}r + \frac{3}{s} - 1 \right) + (t-1)r - (t-2) = \\
 &= r \frac{t-s}{s} - tp - \frac{3t}{s} + 2 \leq \\
 &\leq \left(\frac{ps+1}{s+1} + 2 \right) \frac{t-s}{s} - tp - \frac{3t}{s} + 2 = \\
 &= \frac{-t-1-ps+pt}{s+1} - tp.
 \end{aligned}$$

$\frac{-t-1-ps+pt}{s+1}$ is negative, because the denominator is positive, and the counter is negative, because $-t-1-ps+pt = -t-1-p(s-t)$, where $s-t \geq 0$. Let's continue the deduction:

$$\frac{-t-1-ps+pt}{s+1} - tp < -tp.$$

So,

$$(-t-1)p \leq -tq + (t-1)r - (t-2) < -tp,$$

if we add $(t+1)p$ to this expression, it will be in the $[0, p[$ interval. The conclusion is one step in the algorithm is the following:

$$\begin{aligned}
 (-tq + (t-1)r - (t-2), tr - (t-1), -t) &\xrightarrow{t+1} [(t+1)p - tq + (t-1)r - \\
 &- (t-2)](-t+1)q + tr - (t-1), (t+1)r - t, -(t+1))
 \end{aligned}$$

It can be seen, that the first created expression's value is acceptable, the second one is a specialized case of the generally investigated expression, the only difference is that the value of t is increased by 1. The deduction can be continued this way until $t=s$. The last created line is:

$$[(s+1)p - sq + (s-1)r - (s-2)](-(s+1)q + sr - (s-1), (s+1)r - s, -(s+1))$$

We already now, that the first expression is acceptable, investigating $-(s+1)q + sr - (s-1)$: Lower bound of $-(s+1)q + sr - (s-1)$:

$$\begin{aligned} -(s+1)q + sr - (s-1) &\geq -(s+1)\left(p + r - 1 + \frac{1-p}{s+1}\right) + sr - (s-1) = \\ &= -sp - p - r + s + 1 + p - 1 - s + 1 = -(s+1)p. \end{aligned}$$

Upper bound of $-(s+1)q + sr - (s-1)$:

$$\begin{aligned} -(s+1)q + sr - (s-1) &\leq -(s+1)\left(p + \frac{s-1}{s}r + \frac{3}{s} - 1\right) + sr - (s-1) = \\ &= r\left(\frac{1-s^2}{s} + s\right) - ps - p - 3\frac{s+1}{s} + s + 1 - s + 1 = \frac{r}{s} - ps - p - 1 - \frac{3}{s}. \\ \frac{r}{s} - p - 1 - \frac{3}{s} &< 0, \text{ because} \end{aligned}$$

$$\frac{r}{s} - p - 1 - \frac{3}{s} = \frac{r - ps - s - 3}{s},$$

the denominator is positive, the counter is negative, because the maximum value of r is $p+1$. So, the deduction can be continued:

$$\frac{r}{s} - ps - p - 1 - \frac{3}{s} < -ps.$$

So $-(s+1)p \leq -(s+1)q + sr - (s-1) < -sp$, this means, that adding $(s+1)p$ to this expression the result will be in the acceptable interval.

The next sentence of the algorithm:

$$\xrightarrow{s+1} [(s+1)p - (s+1)q + sr - (s-1)](-(s+1)q + (s+1)r - s, (s+1)r - (s+1), -(s+1))$$

Investigating $-(s+1)q + (s+1)r - s$:

Lower bound of $-(s+1)q + (s+1)r - s$:

$$\begin{aligned} -(s+1)q + (s+1)r - s &\geq -(s+1)\left(p + r - 1 + \frac{1-p}{s+1}\right) + (s+1)r - s = \\ &= (s+1)(1-p) - (1-p) - s = -sp \end{aligned}$$

Upper bound of $-(s+1)q + (s+1)r - s$:

$$-(s+1)q + (s+1)r - s \leq -(s+1)\left(p + \frac{s-1}{s}r + \frac{3}{s} - 1\right) + (s+1)r - s =$$

$$\begin{aligned}
&= r(s+1)\left(\frac{1-s}{s}+1\right) - (s+1)p - 3\frac{s+1}{s} + s+1 - s = (r-3)\frac{s+1}{s} - sp - p + 1 \leq \\
&\leq p - 2 + \frac{p-2}{s} - sp - p + 1.
\end{aligned}$$

$\frac{p-2}{s} - p - 1 < 0$, because

$\frac{p-2}{s} - p - 1 = \frac{p-ps-s-2}{s}$, where the denominator is positive, the counter is negative. So we can continue the deduction:

$$p + \frac{p-2}{s} - sp - p - 1 < p - sp = p(1-s).$$

So in the next step we have to add sp to this expression:

$$\xrightarrow{s} [sp - (s+1)q + (s+1)r - s](-sq + (s+1)r - (s+1), sr - (s+1), -s)$$

Lower bound of $-sq + (s+1)r - (s+1)$:

$$\begin{aligned}
- sq + (s+1)r - (s+1) &\geq -s \left(p + r - 1 + \frac{1-p}{s+1} \right) + (s+1)r - (s+1) = \\
&= -sp - 1 + (p-1)\frac{s}{s+1} + r \geq -sp - 1 + (p-1)\frac{s}{s+1} + \frac{ps+1}{s+1} + 2 = \\
&= -sp + 1 + \frac{2ps-s+1}{s+1} = (1-s)p + \frac{ps-p+2}{s+1} > (1-s)p.
\end{aligned}$$

Upper bound of $-sq + (s+1)r - (s+1)$:

$$\begin{aligned}
- sq + (s+1)r - (s+1) &\leq -s \left(p + \frac{s-1}{s}r + \frac{3}{s} - 1 \right) + (s+1)r - (s+1) = \\
&= 2r - sp - 4 \leq 2p + 2 - sp - 4 = (2-s)p - 2
\end{aligned}$$

So, $(1-s)p < -sq + (s+1)r - (s+1) < (2-s)p - 2$, let's continue the deduction:

$$\xrightarrow{s-1} [(s-1)p - sq + (s+1)r - (s+1)](-(s-1)q + sr - (s+1), (s-1)r - s, -(s-1))$$

Now, instead of investigating the $-(s-1)q + sr - (s+1)$ expression, let's generally investigate the $-(t-1)q + tr - (t+1)$ expression, where $3 \leq t \leq s!$

Lower bound of $-(t-1)q + tr - (t+1)$:

$$\begin{aligned}
-(t-1)q + tr - (t+1) &\geq -(t-1)\left(p + r - 1 + \frac{1-p}{s+1}\right) + tr - (t+1) \geq \\
&\geq -(t-1)\left(p + \frac{1-p}{s+1}\right) + \frac{ps+1}{s+1} + 2 - 2 = \\
&= \frac{-pst + 2ps - t + 2}{s+1} = p(2-t) + \frac{(p-1)(t-2)}{s+1} \geq \\
&\geq p(2-t).
\end{aligned}$$

Upper bound of $-(t-1)q + tr - (t+1)$:

$$\begin{aligned}
 -(t-1)q + tr - (t+1) &\leq -(t-1)\left(p + \frac{s-1}{s}r + \frac{3}{s} - 1\right) + tr - (t+1) \leq \\
 &\leq -tp + p + \frac{3-3t}{s} - 2 + (p+1)\left(1 + \frac{t-1}{s}\right) = \\
 &= (3-t)p + \frac{p(t-s) - p - s + 2(1-t)}{s} < (3-t)p.
 \end{aligned}$$

$p(2-t) \leq -(t-1)q + tr - (t+1) < (3-t)p$, so we have to add $(t-2)p$ to this value to reach the acceptable interval.

$$\begin{aligned}
 &(-(t-1)q + tr - (t+1), (t-1)r - t, -(t-1)) \xrightarrow{t-2} \\
 &[(t-2)p - (t-1)q + tr - (t+1)] \\
 &(-(t-2)q + (t-1)r - t, (t-2)r - (t-1), -(t-2))
 \end{aligned}$$

It can be seen, that the first created expression's value is acceptable, the second one is a specialized case of the generally investigated expression, the only difference is that the value of t is decreased by 1. The deduction can be continued this way until $t=3$. The last created line is:

$$\xrightarrow{1} [p - 2q + 3r - 4](-q + 2r - 3, r - 2, -1)$$

We already now, that the first expression is acceptable, investigating $-q + 2r - 3$:

Lower bound of $-q + 2r - 3$:

$$\begin{aligned}
 -q + 2r - 3 &\geq -p - r + 1 - \frac{1-p}{s+1} + 2r - 3 \geq \\
 &\geq -p + \frac{ps+1}{s+1} + 2 - 2 + \frac{p-1}{s+1} = -p + \frac{ps+p}{s+1} = 0.
 \end{aligned}$$

Upper bound of $-q + 2r - 3$:

$$\begin{aligned}
 -q + 2r - 3 &\leq -p - \frac{s-1}{s}r - \frac{3}{s} + 1 + 2r - 3 \\
 &\leq -p - 2 - \frac{3}{s} + (p+1)\frac{s+1}{s} = -1 + \frac{p-2}{s} \leq p-3.
 \end{aligned}$$

This is acceptable.

$$(-q + 2r - 3, r - 2, -1) \xrightarrow{0} [-q + 2r - 3](r - 2, -1, 0)$$

Investigating $r - 2$!

Lower bound of $r - 2$

$$r - 2 \geq \frac{ps + 1}{s + 1} = p - \frac{p - 1}{s + 1} \geq p - \frac{p - 1}{2} = \frac{p + 1}{2} > 0.$$

Upper bound of $r - 2$:

$$r - 2 \leq p - 1.$$

This is also acceptable.

$$(r - 2, -1, 0) \xrightarrow{0} [r - 2](-1, 0, 0)$$

-1 is negative, so the next line:

$$(-1, 0, 0) \xrightarrow{1} [p - 1](-q, r, -1)$$

Lower bound of $-q$:

$$-q \geq -p - r + 1 - \frac{1 - p}{s + 1} \geq -2p + \frac{p - 1}{s + 1} > -2p.$$

Upper bound of $-q$:

$$\begin{aligned} -q &\leq -p - \frac{s - 1}{s}r - \frac{3}{s} + 1 \leq -p - \frac{s - 1}{s} \left(\frac{ps + 1}{s + 1} + 2 \right) - \frac{3}{s} + 1 = \\ &= -p - \frac{ps(s - 1) + s^2 + 3s}{s(s + 1)} < -p. \end{aligned}$$

We have to add $2p$ to this value:

$$(-q, r, -1) \xrightarrow{2} [2p - q](-2q + r, 2r - 1, -2)$$

The first value is acceptable, the last three values are the starter values. So this is a $2s + 2$ steps cycle with values: $-2q + r$, $2r - 1$ and -2 .

2.6.2. Second triangle-series.

$$P(x) := x^3 - rx^2 + qx - p,$$

$$3 \leq p < +\infty,$$

$$1 \leq s \leq p \text{ (the number of the actual triangle in the series),}$$

$$4 \leq r \leq p + 1,$$

$$p + r - 1 + \frac{1 - p}{s + 1} < q, p - 1 + \frac{s - 1}{s}r + \frac{3}{s} \leq q, q < p - 1 + \frac{s}{s + 1}r + \frac{3}{s + 1}.$$

Let's start "Backward" division process from the $((s + 1)(r - q - 1), (s + 1)(r - 1), -s - 1)$ values. Now we should investigate $(s + 1)(r - q - 1)$: Lower



FIGURE 8. The lower triangle, first triangle-series. Blue: Actually examined points - Yellow: Not semi-CNS polynomials

bound of $(s+1)(r-q-1)$:

$$\begin{aligned}
 (s+1)(r-q-1) &\geq (s+1) \left(r-p+1 - \frac{s}{s+1}r - \frac{3}{s+1} - 1 \right) \geq \\
 &\geq (s+1) \left(-p + \frac{4}{s+1} - \frac{3}{s+1} \right) = \\
 &= -ps - p + 1 = (-s-1)p + 1.
 \end{aligned}$$

Upper bound of $(s+1)(r-q-1)$:

There are two cases:

If $2 + \frac{ps+1}{s+1} < r$:

$$(s+1)(r-q-1) < (s+1) \left(\frac{p-1}{s+1} - p \right) = p-1-sp-p = -1-sp.$$

If $r \leq 2 + \frac{ps+1}{s+1}$:

$$\begin{aligned}
 (s+1)(r-q-1) &\leq (s+1) \left(\frac{r}{s} - p - \frac{3}{s} \right) \leq (s+1) \left(\frac{\frac{ps+1}{s+1} - 1}{s} - p \right) = \\
 &= (s+1) \left(\frac{sp-s}{s(s+1)} - p \right) = p-1-sp-p = -1-sp.
 \end{aligned}$$

So, $(-s-1)p + 1 \leq (s+1)(r-q-1) \leq -sp-1$, we have to add $(s+1)p$ to this expression to reach the acceptable interval. So, here is a single cycle with values: $(s+1)(r-q-1)$, $(s+1)(r-1)$ and $-s-1$.

3. CONCLUSION

We made the whole characterization of the cubic monic semi-CNS polynomials, because a polynomial can be semi-CNS polynomial in the case it is expansive. This condition restricts the area to a triangle. We investigated all the points of the triangle. Comes from the proof that in the set of cubic monic polynomials a polynomial is semi-CNS polynomial, if and only if it is expansive, the constant coefficient is negative, and all other coefficients are non-negative.

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FACULTY OF INFORMATICS, UNIVERSITY OF DEBRECEN

E-mail address: `pvarga@inf.unideb.hu`

MANY-VALUED LOGICS – THEOREM PROVING PROBLEMS

KATALIN PÁSZTOR VARGA AND MAGDA VÁRTERÉSZ

ABSTRACT. The many-valued logics (starting by Post and Łukasiewicz independently in 1921) are perhaps the most natural extensions of the classical logic. Up to the 1990's serious theoretical problems were solved and presented [1]. In the last few years, the development of their applications in computer science, electronics and biology has accelerated.

In this paper, a semantic notion of the consequence relation is given on the basis of results in structural consideration of many-valued logics. This notion could support calculi in many-valued logics semantically. The potential notions for the implication are examined and an implication in compliance with our aims is introduced.

1. INTRODUCTION

The classical propositional logic can be considered as a lattice for any ordering of the two truth values (or for the logical operations $\wedge$ and $\vee$). The base of a semantic notion of consequence is just that the propositional logic is a Boolean algebra and the implication satisfies the identity $x \supset y = \neg x \vee y$. This guarantees the applicability of modus ponens. Let $U_n = \{0, 1, \dots, n-1\}$ be the set of truth values. With natural ordering, U_n is a lattice, therefore min and max are the adequate extensions of the classical operations $\wedge$ and $\vee$. The extension of $\neg$ keeps only its some formal properties in such a way that the obtained structure is closed to a Boolean algebra. For this reason some calculi (for example Gentzen type ones) can be defined for theorem proving. Our results obtained in the course of examination of some definitions of implication open the door to introduce a semantic notion of consequence and formulate a semantic decision problem.

2. ABOUT PROPOSITIONAL LOGICS

According to the widespread treatment, the logic language describes an algebraic structure which is the basis of semantics. We connect the (syntactic

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or semantic) notion of consequence to this. Often, the notion of consequence appears as a consequence operator. One tool of the method is the logical matrix method. This method runs to the intuitionistic logic, natural deduction and Gentzen's method of sequents, however it opens the way for the theorem proving by a semantic notion of the consequence.

2.1. Logical matrices. Let J be any set. By a signature Ω we understand any function from J into N ; the notation is $\Omega = (n_j)_{j \in J}$. By an Ω -algebra we mean a pair $\langle U, \{o_j | j \in J\} \rangle$, where $U \neq \emptyset$ is a set, called universe, and each o_j is an n_j -argument operation in U , namely $o_j : U^{n_j} \rightarrow U$.

We consider an arbitrary logic language $L = \langle V, Con, F \rangle$, where V is a set of propositional variables; Con is the set of logical connectives; F is the set of formulas generated by V and Con in a standard way.

Let $J = \{1, 2, \dots, |Con|\}$, $\Omega = (arg(o_j))_{j \in J}$, $o_j \in Con$. F can be regarded as the universe of an Ω -algebra with operations induced by connectives from Con . This algebra is freely generated by V .

Let Ω be the signature of a logic language algebra L . A logical matrix M for L is a triple

$$\langle U, \{o_j | j \in J\}, U^* \rangle,$$

where $\langle U, \{o_j | j \in J\} \rangle$ is an Ω -algebra, U^* is a nonempty subset of U . A logical matrix

$$\langle W, \{o_j | j \in J\}, W^* \rangle$$

is a submatrix of the matrix M if $W \subseteq U$, W is closed with respect to each operation o_j for $j \in J$ and $W^* = U^* \cap W$.

After this, let's define the semantics as an interpretation over a logical matrix M for the language L . Every function $I : V \rightarrow U$ (called an interpretation of L over M) uniquely can be extended to a homomorphism (called a valuation of formulas) from the set of formulas F to the universe U :

- (a) $|p|_I = I(p)$ for $p \in V$;
- (b) $|o_j(\alpha_1, \dots, \alpha_{n_j})|_I = o_j(|\alpha_1|_I, \dots, |\alpha_{n_j}|_I)$, for every n_j -ary connective (operation) o_j and for all $\alpha_1, \dots, \alpha_{n_j} \in F$.

A formula α of a logic language L is satisfiable over a logical matrix M if there exists an interpretation I such that $|\alpha|_I \in U^*$. α is a tautology over M , when $|\alpha|_I \in U^*$ for every I . The set of all tautologies over M will be denoted by $E(M)$. If a logical matrix N is a submatrix of M , then $E(M) \subseteq E(N)$.

2.2. Definitions of propositional calculi. The construction of propositional calculi can follow several methodological ways.

The algebraic method: is fundamental and also prior to other ones.

By a propositional logic we mean a quadruple $\langle L, M, In, Pr \rangle$, where

- (a) $L = \langle V, Con, F \rangle$ is a logic language algebra;

- (b) $M = \langle U, \{o_j \mid j \in J\}, U^* \rangle$ is a logical matrix for L ;
- (c) $In = \{I \mid I : V \rightarrow U \text{ is an interpretation of } L\}$;
- (d) $Pr : \mathcal{P}(F) \rightarrow \mathcal{P}(F)$ is a consequence operation providing a mechanism for acceptance or refutation of formulas. ($\mathcal{P}(F)$ is the power set of F .)

Definition 1. *The consequence operation Pr is an operation which satisfies the following conditions for any $X, Y \subseteq F$ and $\alpha, \beta \in F$:*

- (1) $X \subseteq Pr(X) \subseteq F$;
- (2) if $X \subseteq Pr(Y)$ then $Pr(X) \subseteq Pr(Y)$;
- (3) if $\alpha \in Pr(X)$ then there exists a finite set Y such that $Y \subseteq X$ and $\alpha \in Pr(Y)$;
- (4) $Pr(Pr(X)) \subseteq Pr(X)$.

The axiomatic method:: Informally, a propositional calculus is axiomatically given if

- (a) its language algebra $L = \langle V, Con, F \rangle$ is specified;
- (b) it is fixed a finite set of formulas (axiom system);
- (c) it is specified a finite set of rules $r_1, \dots, r_m$ for the acceptance of formulas from among previously accepted ones and the axioms.

Definition 2. *A rule over F is a nonempty relation*

$$r = \{(\alpha_1, \dots, \alpha_n, \alpha) \mid \alpha_1, \dots, \alpha_n, \alpha \in F\}.$$

Definition 3. *A proof is a finite chain of formulas $\alpha_1, \dots, \alpha_k$ such that for each i ($1 \leq i \leq k$),*

- either α_i is an axiom
- or there exist indices $i_1, \dots, i_l$ smaller than i and such that $(\alpha_{i_1}, \dots, \alpha_{i_l}, \alpha_i) \in r_j$ for a certain rule $r_j \in r_1, \dots, r_m$.

Definition 4. *Let X be any set of formulas. A formula α is derived from X if*

- there is a chain of formulas $\alpha_1, \dots, \alpha_k$ such that $\alpha_k = \alpha$ and,
- for each i ($1 \leq i \leq k$), either $\alpha_i \in X$ or there exist indices $i_1, \dots, i_l$ smaller than i and such that $(\alpha_{i_1}, \dots, \alpha_{i_l}, \alpha_i) \in r_j$ for a certain $r_j \in r_1, \dots, r_m$.

$Pr^* : X \rightarrow \{\alpha \mid \alpha \text{ is derived from } X\}$ satisfies conditions called for a consequence operation. A logical matrix M for a language L is said to be strongly adequate for the calculus if the consequence operation coincides with Pr^* .

Let U_n be the set of truth values $\{0, 1, 2, \dots, n-1\}$ ($n \geq 2$). Moreover let $0 \leq S \leq n-1$. $0, 1, \dots, S$ are called non-designated truth values, and $U_n^* = S+1, \dots, n-1$ are the designated ones. In the classical logic the set of

truth values is $U_2 = \{0, 1\}$, where 1 is the true (designated) truth value. By a classical two-valued propositional logic we mean a quadruple $\langle L, M, In, Pr \rangle$, where

- (a) $L = \langle V, \{\neg, \wedge, \vee\}, F \rangle$ is the language algebra with signature $(1, 2, 2)$;
- (b) $M = \langle U_2, \{\neg, \wedge, \vee\}, \{1\} \rangle$ is a logical matrix with Boolean algebra $\langle U_2, \{\neg, \wedge, \vee\} \rangle$ (the operations in it are $\wedge, \vee$ as lattice operations, and the complementation operation is $\neg$);
- (c) $In = \{I \mid I : V \rightarrow U_2 \text{ is an interpretation of } L\}$;
- (d) $Pr : \mathcal{P}(F) \rightarrow \mathcal{P}(F)$ is the usual semantic consequence operation ($\alpha \in Pr(X)$ if and only if α has the truth value 1, when X).

M is strongly adequate for the classical propositional calculus (axioms, modus ponens) [2].

In the same way, by a propositional many-valued logic we would like to mean a quadruple $\langle L, M, In, Pr \rangle$ where

- (a) $L = \langle V, Con, F \rangle$ is a language algebra;
- (b) $M = \langle U_n, Op, U_n^* \rangle$ is a logical matrix for L ;
- (c) $In = \{I \mid I : V \rightarrow U_n \text{ is an interpretation of } L\}$;
- (d) $Pr : \mathcal{P}(F) \rightarrow \mathcal{P}(F)$ is an operation of consequence.

For the many-valued logic, we look for such algebras which keep the most important properties of the Boolean algebra and we look for a consequence operation.

2.3. A possible definition for the semantic consequence. Let $L = \langle V, Con, F \rangle$ be a language algebra and let $M = \langle U_n, Op, U_n^* \rangle$ be a logical matrix for the language L .

Definition 5. A formula α is a semantical S -consequence of formulas $\alpha_1, \dots, \alpha_k$, denoted as

$$\alpha_1, \dots, \alpha_k \models_S \alpha,$$

if for any interpretation in which the truth value of every formula α_i ($1 \leq i \leq k$) is designated, the truth value of α is also designated with at least the same truth value as the maximum of the truth values of formulas $\alpha_1, \dots, \alpha_k$ in the underlying interpretation.

In classical logic, the notion of the consequence leads to the decision problem through the deduction theorem. The deduction theorem requires the classical syllogism "modus ponens".

The modus ponens in a many-valued logic is valid if $\alpha \supset \beta, \alpha \models_S \beta$, i.e. if $\alpha \supset \beta$ and α have designated values then β has a designated value too. For this it would be necessary such a negation ($\neg$) which changes the designated status of a truth value to non-designated and vice versa; moreover $x \supset y = \neg x \vee y$ by definition.

2.4. Potential algebras for the many-valued logic. It is well-known that the set U_n ($n > 2$) (with natural ordering) is a distributive (non-complemented) lattice with operations $\min$ ($\wedge$) and $\max$ ($\vee$). Now we should define a unary operation $-$ such that for all $x, y \in U_n$

$$-(-x) = x \text{ and } -(x \wedge y) = (-x) \vee (-y).$$

Possible such structures are:

- (1) the quasi-Boolean algebra

$$\langle U_n, \{\wedge, \vee, -, 0, (n-1)\} \rangle,$$

where $\langle U_n, \{\wedge, \vee\} \rangle$ is the distributive lattice above, with lattice zero 0, unit $n-1$. The properties of the unary operation $-$ imply that

$$-0 = (n-1), -(n-1) = 0 \text{ and } -(x \vee y) = (-x) \wedge (-y).$$

- (2) the symmetric Heyting algebra

$$\langle U_n, \{\wedge, \vee, -, \Rightarrow, 0, (n-1)\} \rangle,$$

where $\langle U_n, \{\wedge, \vee, -, 0, (n-1)\} \rangle$ is a quasi-Boolean algebra such that

$$(x \Rightarrow -y) = (y \Rightarrow -x), ((x \Rightarrow -y) \Rightarrow -x) = y \text{ and}$$

$$((x \Rightarrow y) \Rightarrow z) = -((-x \Rightarrow z) \Rightarrow -(y \Rightarrow z))$$

for all $x, y, z \in U_n$.

- (3) the Łukasiewicz algebra

$$\langle U_n, \{\wedge, \vee, \sim, \Rightarrow, s_0, \dots, s_{n-1}, 0, (n-1)\} \rangle,$$

where $\langle U_n, \{\wedge, \vee, \sim, \Rightarrow, 0, (n-1)\} \rangle$ is a symmetric Heyting algebra and the operations $s_0, \dots, s_{n-1}$ satisfy some (here not detailed) axioms [1].

The following operations defined by Post and Łukasiewicz are the extensions of the operations of the classical logic. Some possible extensions are:

- (a) Łukasiewicz negation $\tilde{-}x \Leftrightarrow n-1-x$
- (b) Łukasiewicz implication $x_1 \tilde{\Rightarrow} x_2 \Leftrightarrow \begin{cases} n-1 & \text{if } x_1 \leq x_2 \\ (n-1) - x_1 + x_2 & \text{if } x_1 > x_2 \end{cases}$
- (aa) Post negation $\bar{-}x \Leftrightarrow (x+1) \bmod n$
- (bb) Post implication $x_1 \bar{\Rightarrow} x_2 \Leftrightarrow \begin{cases} n-1 & \text{if } x_1 \leq x_2 \\ x_2 & \text{if } x_1 > x_2, S < x_1 \\ (n-1) - x_1 + x_2 & \text{if } x_1 > x_2, S \geq x_1 \end{cases}$

By the Post/Łukasiewicz operations we obtain some foregoing structures:

- (1) $\langle U_n, \{\wedge, \vee, \tilde{-}, 0, (n-1)\} \rangle$ is a quasi-Boolean algebra.
- (2) $\langle U_n, \{\wedge, \vee, \tilde{-}, \tilde{\Rightarrow}, 0, (n-1)\} \rangle$ is a symmetric Heyting algebra.

(3) To have a Łukasiewicz algebra

$$\langle U_n, \{\wedge, \vee, \tilde{\neg}, \tilde{\supset}, j_0, \dots, j_{n-1}, 0, (n-1)\} \rangle$$

we introduce some unary operations in U_n :

$$j_i x \Leftrightarrow \begin{cases} n-1 & \text{if } x = i, \\ 0 & \text{if } x \neq i. \end{cases}$$

3. PROBLEMS WITH n -VALUED OPERATIONS

Łukasiewicz implication is defined by

$$x_1 \tilde{\supset} x_2 \Leftrightarrow \begin{cases} n-1 & \text{if } x_1 \leq x_2 \\ (n-1) - x_1 + x_2 & \text{if } x_1 > x_2 \end{cases}$$

or by the Table 1. Designated values are marked by an asterisk.

$\tilde{\supset}$	0	1	...	$S-1$	S	$S+1^*$	...	$n-3^*$	$n-2^*$	$n-1^*$
0	$n-1$	$n-1$	...	$n-1$	$n-1$	$n-1$	...	$n-1$	$n-1$	$n-1$
1	$n-2$	$n-1$	...	$n-1$	$n-1$	$n-1$	...	$n-1$	$n-1$	$n-1$
2	$n-3$	$n-2$	...	$n-1$	$n-1$	$n-1$	...	$n-1$	$n-1$	$n-1$
$\vdots$			...				...			
$S-1$	$n-S$	$n-S+1$	...	$n-1$	$n-1$	$n-1$	...	$n-1$	$n-1$	$n-1$
S	$n-S-1$	$n-S$	...	$n-2$	$n-1$	$n-1$	...	$n-1$	$n-1$	$n-1$
$S+1^*$	$n-S-2$	$n-S-1$	...	$n-3$	$n-2$	$n-1$	...	$n-1$	$n-1$	$n-1$
$\vdots$			...				...			
$n-3^*$	2	3	...	$S+1$	$S+2$	$S+3$	...	$n-1$	$n-1$	$n-1$
$n-2^*$	1	2	...	S	$S+1$	$S+2$	...	$n-2$	$n-1$	$n-1$
$n-1^*$	0	1	...	$S-1$	S	$S+1$	...	$n-3$	$n-2$	$n-1$

TABLE 1. Łukasiewicz implication

We can see, that in the case where $n = 3$ and $S = 0$, $S+1 \tilde{\supset} S$ and $S+1$ are designated, but S is not.

Post implication is defined by

$$x_1 \bar{\supset} x_2 \Leftrightarrow \begin{cases} n-1 & \text{if } x_1 \leq x_2 \\ x_2 & \text{if } x_1 > x_2, S < x_1 \\ (n-1) - x_1 + x_2 & \text{if } x_1 > x_2, S \geq x_1 \end{cases}$$

or by the Table 2. This implication is semantically good for the modus ponens. Whether it can be written in form the $\neg x \vee y$?

The Łukasiewicz and Post negations satisfy some algebraic properties, but they are not convenient for the semantic, because the negation of a designated value is not non-designated in every case. If $n = 3$, $S = 1$ and a formula has 1 truth value, then it is non-designated. In the same case, the Łukasiewicz negation of the formula has

$$\tilde{\neg} 1 = 3 - 1 - 1 = 1$$

$\supset$	0	1	...	$S-1$	S	$S+1^*$	...	$n-3^*$	$n-2^*$	$n-1^*$
0	$n-1$	$n-1$	...	$n-1$	$n-1$	$n-1$	...	$n-1$	$n-1$	$n-1$
1	$n-2$	$n-1$	...	$n-1$	$n-1$	$n-1$	...	$n-1$	$n-1$	$n-1$
2	$n-3$	$n-2$	...	$n-1$	$n-1$	$n-1$	...	$n-1$	$n-1$	$n-1$
...			...				...			
$S-1$	$n-S$	$n-S+1$	...	$n-1$	$n-1$	$n-1$	...	$n-1$	$n-1$	$n-1$
S	0	1	...	$S-1$	$n-1$	$n-1$	...	$n-1$	$n-1$	$n-1$
$S+1^*$	0	1	...	$S-1$	S	$n-1$	...	$n-1$	$n-1$	$n-1$
...			...				...			
$n-3^*$	0	1	...	$S-1$	S	$S+1$	...	$n-1$	$n-1$	$n-1$
$n-2^*$	0	1	...	$S-1$	S	$S+1$	...	$n-3$	$n-1$	$n-1$
$n-1^*$	0	1	...	$S-1$	S	$S+1$	...	$n-3$	$n-2$	$n-1$

TABLE 2. Post implication

truth value, so it is non-designated too.

4. SUITABLE NEGATION AND IMPLICATION FOR THE SEMANTIC CONSEQUENCE

A new negation and implication can be defined:

$$\hat{\neg}x \Rightarrow \bigvee_{i=0}^S j_i(x),$$

$$x_1 \hat{\supset} x_2 \Rightarrow \max_{0 \leq i \leq S} j_i(x_1) \vee x_2.$$

$\hat{\supset}$	0	1	...	$S-1$	S	$S+1^*$	...	$n-3^*$	$n-2^*$	$n-1^*$
0	$n-1$	$n-1$	...	$n-1$	$n-1$	$n-1$	...	$n-1$	$n-1$	$n-1$
1	$n-1$	$n-1$	...	$n-1$	$n-1$	$n-1$	...	$n-1$	$n-1$	$n-1$
2	$n-1$	$n-1$	...	$n-1$	$n-1$	$n-1$	...	$n-1$	$n-1$	$n-1$
...			...				...			
$S-1$	$n-1$	$n-1$	...	$n-1$	$n-1$	$n-1$	...	$n-1$	$n-1$	$n-1$
S	$n-1$	$n-1$	...	$n-1$	$n-1$	$n-1$	...	$n-1$	$n-1$	$n-1$
$S+1^*$	0	1	...	$S-1$	S	$S+1$	...	$n-3$	$n-2$	$n-1$
...			...				...			
$n-3^*$	0	1	...	$S-1$	S	$S+1$	...	$n-3$	$n-2$	$n-1$
$n-2^*$	0	1	...	$S-1$	S	$S+1$	...	$n-3$	$n-2$	$n-1$
$n-1^*$	0	1	...	$S-1$	S	$S+1$	...	$n-3$	$n-2$	$n-1$

TABLE 3. The new implication

Firstly, this implication satisfies the identity

$$x_1 \hat{\supset} x_2 = \hat{\neg}x_1 \vee x_2.$$

Secondly, we can show that the notion of the consequence defined on the basis of the new tools is conform with the classical case.

Let every formula α_i ($1 \leq i \leq m$) have a designated value under the interpretation I and $|\alpha_1|_I = \max_{1 \leq j \leq m} |\alpha_j|_I$, moreover let

$$\begin{aligned} |\alpha_k \hat{\vee} \dots \hat{\vee} \alpha_m \hat{\vee} \beta|_I &= |\hat{\vee} \alpha_k \vee \dots \vee \hat{\vee} \alpha_m \vee \beta|_I = \\ &= \max \{|\hat{\vee} \alpha_k|_I, \dots, |\hat{\vee} \alpha_m|_I, |\beta|_I\}. \end{aligned}$$

Because each α_j has a designated value for I , $|\hat{\vee} \alpha_j|_I = 0$,

$$\max \{|\hat{\vee} \alpha_k|_I, \dots, |\hat{\vee} \alpha_m|_I, |\beta|_I\} = |\beta|_I.$$

Let $\alpha_1, \dots, \alpha_m \models_S \beta$. In the $(m-1)$ -th step, we have $\alpha_1 \models_S \alpha_2 \hat{\vee} \dots \hat{\vee} \alpha_m \hat{\vee} \beta$, where in our case it must be $|\alpha_1|_I \leq |\alpha_2 \hat{\vee} \dots \hat{\vee} \alpha_m \hat{\vee} \beta|_I = |\beta|_I$. So β must have a designated value with at least the same truth value in I as the greatest designated value from the formulas $\alpha_1, \dots, \alpha_m$.

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DEPARTMENT OF PROGRAMMING LANGUAGES AND COMPILERS, EÖTVÖS LORÁND UNIVERSITY

E-mail address: pkata@ludens.elte.hu

DEPARTMENT OF COMPUTER SCIENCE, DEBRECEN UNIVERSITY

E-mail address: varteresz.magda@inf.unideb.hu

AN APPLICATION OF TIME SERIES ANALYSIS IN FINANCE

JITKA ZICHOVÁ

ABSTRACT. Time series is a sequence of observations of a variable ordered in time. The contribution is devoted to some demonstrations of linear time series models in financial data analysis. The attention is focused in the problem of parameter estimation and predicting future behaviour of the recorded series. We can be interested in forecasting exchange rates, stock prices, stock exchange indices or other financial variables. Univariate and multivariate ARMA models represent a useful tool for this purpose. The simplest is the first order autoregression AR(1) defined by the equation

$$X_t = bX_{t-1} + e_t,$$

where X_t is the value of the series at time t and e_t is a random error sequence.

Simulation experiments were performed by several authors to study the properties of the estimates but applications to real data sets were published rarely. That is why we introduce a presentation of the mentioned estimation procedures in selected financial time series models and their comparison with methods implemented in software products. Further, the prediction capability of studied models was checked which will be shown in graphical outputs.

1. INTRODUCTION

Time series is a sequence of observations of a variable ordered in time. This paper is devoted to the application of linear time series models in financial data analysis. The attention is focused in the problem of parameter estimation and predicting future behaviour of the recorded series. We are usually interested in forecasting exchange rates, stock prices, stock exchange indices or other financial variables. Univariate and multivariate ARMA models (see e.g. [6], [8]) represent a useful tool for this purpose.

We show an estimation methodology which was proposed especially for non-negative time series and we generalize some known results to a new type of multivariate models, namely the second order autoregression. The behaviour

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of the estimates will be investigated in a simulation study as well as in real data analysis. At the end, some prediction results will be presented.

2. ESTIMATION IN NON-NEGATIVE AUTOREGRESSIVE PROCESSES

A simple linear time series model is first order autoregression AR(1) defined by the equation

$$X_t = bX_{t-1} + e_t, \quad t = 1, 2, \dots$$

where X_t is the value of the series at time t and e_t is an error sequence called white noise which is created by uncorrelated random variables. The condition $-1 < b < 1$ guarantees the stationarity of the AR(1) model ([8], p. 33). We would like to estimate the parameter b to be able to use the estimated model for the description of the observed series and the prediction of its future development. Traditional estimation procedures such as maximum likelihood method or least squares method are described in literature ([6], [8]) and implemented in different types of statistical software. They are based on the assumption of normal probability distribution and zero mean of the white noise e_t . Nevertheless, in practice we often observe non-negative time series which do not satisfy these assumptions.

Therefore Bell and Smith in [5] introduced a special approach for parameter estimation. They supposed AR(1) with non-negative white noise having a non-specified continuous probability distribution with a distribution function F , which satisfies $F(d) - F(c) < 1$ for all $0 < c < d < \infty$, and $0 < b < 1$. They defined a simple estimate of the parameter b based on the observed series of length n in the form

$$b^* = \min \left(\frac{X_2}{X_1}, \frac{X_3}{X_2}, \dots, \frac{X_n}{X_{n-1}} \right).$$

It was showed that b^* has very good small sample behaviour and convergence properties (see [5]). The white noise e_t is e.g. the exponentially distributed one with mean λ studied by Anděl in [2]. He proved that b^* can be obtained by maximizing the conditional likelihood function of the variables $X_2, X_3, \dots, X_n$ having fixed X_1 which is given by

$$\lambda^{-n+1} \exp \left(- \sum_{t=2}^n \frac{X_t - bX_{t-1}}{\lambda} \right)$$

under the non-negativity conditions of the white noise e_t

$$X_t - bX_{t-1} \geq 0, \quad t = 2, 3, \dots, n.$$

Later on, this approach, which we call method of linear programming, was generalized to other univariate models (see [2], [1], [4]) and to multivariate

AR(1) model ([3]). The last mentioned model is given by the equation

$$X_t = UX_{t-1} + e_t,$$

where X_t is a d -dimensional vector of observed variables at time t and e_t is white noise of the same dimension having independent components. The parameter matrix U with elements $u_{ij} \geq 0, i, j = 1, \dots, d$ is supposed to have all eigenvalues inside the unit circle which ensures the stationarity of the process X_t . The theory of multivariate autoregressions, their properties and stationarity conditions can be found in [8].

Simulation experiments were performed to study the properties of the linear programming estimates. In all models, the quality of the estimates of this type was satisfactory even if the probability distribution of the white noise was not exponential one on which is based the maximization procedure. Some theoretical results concerning the rate of convergence of the estimates were derived. Applications to real data sets, especially from finance, were published rarely. That is why an application of univariate and multivariate AR(1) to an exchange rate data set was performed in [9]. We also applied a multivariate second order autoregression to the same data and we show the results in the following text.

3. MULTIVARIATE AR(2) MODEL

Multivariate second order autoregression is defined as follows

$$X_t = UX_{t-1} + VX_{t-2} + e_t.$$

We assume non-negative white noise and non-negative elements $u_{ij}, v_{ij}, i, j = 1, 2, \dots, d$ of the parameter matrices U, V . The stationarity is satisfied if all roots of the determinant $|\lambda^2 I - \lambda U - V|$ lie inside the unit circle where I is the d -dimensional identity matrix. Further, let us suppose $d=2$. If the independent components of the white noise e_t are exponentially distributed with means λ_1, λ_2 respectively, the conditional likelihood function of $X_3, \dots, X_n$ given X_1, X_2 is

$$\begin{aligned} & \lambda_1^{-n+2} \exp \left(\frac{-1}{\lambda_1} \sum_{t=3}^n (X_{t1} - u_{11}X_{t-1,1} - u_{12}X_{t-1,2} - v_{11}X_{t-2,1} - v_{12}X_{t-2,2}) \right) \\ & \times \lambda_2^{-n+2} \exp \left(\frac{-1}{\lambda_2} \sum_{t=3}^n (X_{t2} - u_{21}X_{t-1,1} - u_{22}X_{t-1,2} - v_{21}X_{t-2,1} - v_{22}X_{t-2,2}) \right). \end{aligned}$$

We obtain the estimates of the parameters $u_{ij}, v_{ij}, i, j = 1, 2$ by maximization of this function under the conditions

$$\begin{aligned} X_{t1} - u_{11}X_{t-1,1} - u_{12}X_{t-1,2} - v_{11}X_{t-2,1} - v_{12}X_{t-2,2} &\geq 0, \quad t = 3, \dots, n \\ X_{t2} - u_{21}X_{t-1,1} - u_{22}X_{t-1,2} - v_{21}X_{t-2,1} - v_{22}X_{t-2,2} &\geq 0, \quad t = 3, \dots, n \end{aligned}$$

$$u_{ij} \geq 0, v_{ij} \geq 0, \quad i, j = 1, 2.$$

Using the same argumentation as in [3] we can solve the linear programming problem of maximizing

$$(u_{11} + v_{11}) \frac{1}{n} \sum_{t=1}^n X_{t,1} + (u_{12} + v_{12}) \frac{1}{n} \sum_{t=1}^n X_{t,2},$$

$$(u_{21} + v_{21}) \frac{1}{n} \sum_{t=1}^n X_{t,1} + (u_{22} + v_{22}) \frac{1}{n} \sum_{t=1}^n X_{t,2}$$

under the mentioned conditions giving almost the same solution as the original maximization if the sample size n is sufficiently large.

4. SIMULATION STUDY

A simulation experiment demonstrated a satisfactory behaviour of the linear programming estimates. We generated the two-dimensional AR(2) process

$$X_t = UX_{t-1} + VX_{t-2} + e_t$$

with the matrices

$$U = \begin{pmatrix} 0.3 & 0.05 \\ 0.4 & 0.1 \end{pmatrix}$$

and

$$V = \begin{pmatrix} 0.25 & 0.15 \\ 0.2 & 0.45 \end{pmatrix}.$$

The process e_t was simulated as a sequence of independent random vectors with independent exponentially distributed components. The mean of the exponential distribution was chosen equal to 1 for both components of the white noise. One hundred series of length n equal to 20, 50 and 100 were generated. The averages of the estimated elements of the matrices U , V and sample standard deviations in parentheses are summarized in the following tables. A very good rate of convergence of the estimates can be observed even for the moderate length of the series.

Table 1. The estimates of the elements of U

n	u_{11}^*	u_{12}^*	u_{21}^*	u_{22}^*
20	0.321 (0.139)	0.087 (0.097)	0.437 (0.161)	0.122 (0.123)
50	0.319 (0.066)	0.055 (0.045)	0.411 (0.066)	0.111 (0.061)
100	0.311 (0.033)	0.053 (0.026)	0.410 (0.028)	0.099 (0.024)

Table 2. The estimates of the elements of V

n	v_{11}^*	v_{12}^*	v_{21}^*	v_{22}^*
20	0.205 (0.138)	0.142 (0.101)	0.232 (0.195)	0.402 (0.121)
50	0.247 (0.076)	0.140 (0.041)	0.204 (0.078)	0.435 (0.055)
100	0.249 (0.029)	0.144 (0.027)	0.198 (0.029)	0.449 (0.019)

5. DATA ANALYSIS

The results for two data sets of exchange rates, CZK/EUR and CZK/USD, are presented. Both series contain 86 monthly average exchange rates observed from January 1999, when EURO was introduced, to February 2006 and published by Czech National Bank (see [9] for details). The means are 32.884 for CZK/EUR and 31.502 for CZK/USD. We can see the graph of both series in figure 1. The dollar series exhibits larger variability in time. The time values on the horizontal axis are numbers of months

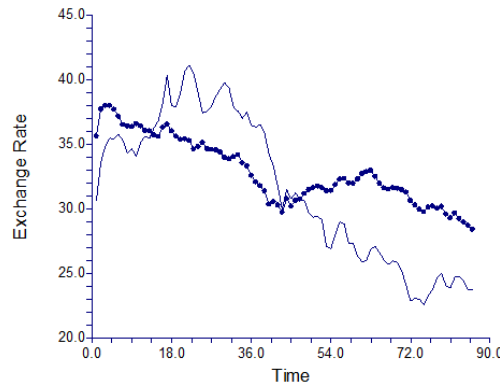


FIGURE 1. Series CZK/EUR – dots and CZK/USD – line

The data were subjected to logarithmic transformation. In our case, the variance was reduced and the nonnegativity of the series was conserved. The method of linear programming was applied to estimate the parameter matrices of the two-dimensional AR(2) for the transformed data. The results are

$$U^* = \begin{pmatrix} 0.9903 & 0 \\ 0 & 0.9783 \end{pmatrix}$$

and V^* having all elements equal to zero. This means, according to the equation of the multivariate AR(2), the procedure identified two separate univariate autoregressions of first order. The estimated parameters are $b_{EUR}^*=0.9903$ in the AR(1) for CZK/EUR and $b_{USD}^*=0.9783$ in the AR(1) for CZK/USD. For comparison, two-dimensional AR(2) was estimated from the logarithmic series using the library Time Series Pack of the software system *Mathematica* where an algorithm based on the classical maximum likelihood approach with the normality assumption is implemented (see [7], [8]). We got the following estimates of the parameter matrices:

$$\hat{U} = \begin{pmatrix} 1.1926 & -0.0704 \\ -0.0489 & 1.2357 \end{pmatrix},$$

$$\hat{V} = \begin{pmatrix} -0.1890 & 0.0659 \\ 0.0703 & -0.2584 \end{pmatrix}.$$

The results are similar to those obtained by the linear programming method. The estimate of the matrix U has the diagonal elements close to one and the off diagonal elements close to zero, the estimate of V has small off diagonal elements.

The model obtained by *Mathematica* was used for predicting an unknown future. For this purpose, both observed series had been shortened so that they contained only 74 observations from January 1999 to February 2005. The development of the series in the time period March 2005 - February 2006 was forecasted from the reduced data. To show the quality of the application of a multivariate model of higher order we compare in figure 2 future forecasts obtained by three models estimated in *Mathematica*. The observed data are depicted by line, the forecasts are represented by triangles in univariate AR(1) model, by squares in multivariate AR(1) and by circles in our multivariate AR(2). The values of estimated parameters in univariate and multivariate AR(1) can be found in [9].

Another possibility of studying the quality of the models presented in figure 2 is a comparison of root mean square of prediction errors which was calculated from the observed values X_t and their forecasts $\hat{X}_t, t = 75, \dots, 86$ constructed using the shortened series as

$$RMS = \sqrt{\frac{1}{12} \sum_{t=75}^{86} (X_t - \hat{X}_t)^2}.$$

The values of RMS are summarized in table 3.

Table 3. RMS of forecasted series

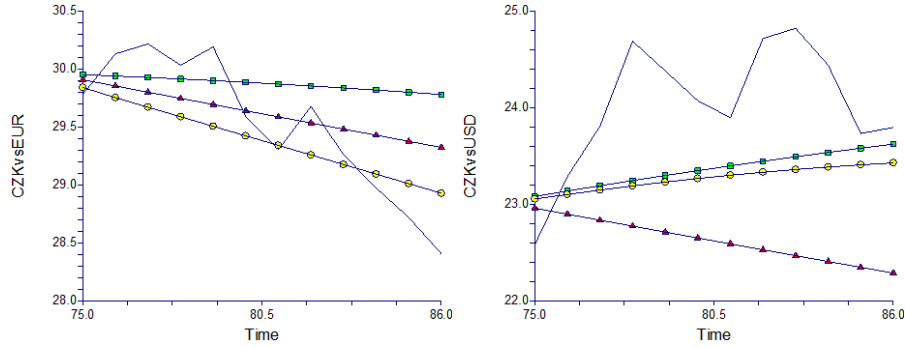


FIGURE 2. Forecasting in the series CZK/EUR – left and CZK/USD – right

Data	Uni AR(1)	Mult AR(1)	Mult AR(2)
CZK/EUR	0.430	0.630	0.375
CZK/USD	1.649	0.949	1.026

We can see from figure 2 and table 3 that in the series CZK/EUR, the forecasts in all models are similar. The application of the multivariate AR(2) seems to have given best fit to the actual development of this exchange rate. In the case of the series CZK/USD with larger variability in time, the univariate model predicted continuation of globally decreasing trend meanwhile the multivariate ones gave almost identical predicted increasing trend which is in accordance with the actual locally increasing part of the series. The values in the forecasted time horizon were underestimated.

6. CONCLUSIONS

At the end, let us make some conclusions which can be drawn from our work. First, we introduced the linear programming estimation procedure for the two-dimensional second order autoregression. It is a generalization of the results published in [3]. Simulation study showed good convergence rate of the estimates to the true parameter values.

The multivariate AR(2) model was then used in the analysis of two exchange rate series. The estimates of parameter matrices were constructed using the linear programming approach and by means of an estimation algorithm implemented in *Mathematica* which is based on standard assumptions. We obtained similar values of the estimates. The small off diagonal elements of parameter matrices indicate only a weak dependence of the value of one exchange rate at time t on past values of the other exchange rate at times

$t - 1$, $t - 2$. The linear programming method did not confirm the dependence on the values at time $t - 2$ having identified AR(1) model.

The software product *Mathematica* was used to study the prediction capability of three autoregressive models in our data set. We were concerned in forecasting an unknown future so that we separated the last observed 12 months from both series and constructed forecasts using shortened data of length 74. We compared the predictions with the observed values $X_{75}, \dots, X_{86}$. Both the graphical output and the values of RMS indicate that the forecasting quality of the models included in our study is good for the more stable series CZK/EUR and the results depend on the type of the model in the case of the series CZK/USD. This fact can be regarded as a consequence of larger variability of the series. It seems that we can recommend using multivariate second order autoregression for data analysis of this type because this model works with more information and therefore leads to better prediction results.

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DEPARTMENT OF PROBABILITY AND STATISTICS, FACULTY OF MATHEMATICS AND PHYSICS,
CHARLES UNIVERSITY PRAGUE

E-mail address: `zichova@karlin.mff.cuni.cz`