

ON SERIES OF TRANSLATES OF POSITIVE FUNCTIONS. III

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Dedicated to the memory of Jean-Pierre Kahane

Abstract. Suppose Λ is a discrete infinite set of nonnegative real numbers. We say that Λ is of type 1 if the series $s(x) = \sum_{\lambda \in \Lambda} f(x + \lambda)$ satisfies a zero-one law. This means that for any non-negative measurable $f: \mathbb{R} \rightarrow [0, +\infty)$ either the convergence set $C(f, \Lambda) = \{x : s(x) < +\infty\} = \mathbb{R}$ modulo sets of Lebesgue zero, or its complement the divergence set $D(f, \Lambda) = \{x : s(x) = +\infty\} = \mathbb{R}$ modulo sets of measure zero. If Λ is not of type 1 we say that Λ is of type 2.

We show that there is a universal Λ with gaps monotone decreasingly converging to zero such that for any open subset $G \subset \mathbb{R}$ one can find a characteristic function f_G such that $G \subset D(f_G, \Lambda)$ and $C(f_G, \Lambda) = \mathbb{R} \setminus G$ modulo sets of measure zero. We also consider the question whether $C(f, \Lambda)$ can contain non-degenerate intervals for continuous functions when $D(f, \Lambda)$ is of positive measure.

The above results answer some questions raised in a paper of Z. Buczolich, J.-P. Kahane, and D. Mauldin.

1. Introduction

This paper was written for the Kahane memorial volume of *Analysis Mathematica*. We selected a topic related to Jean-Pierre Kahane's work and decided to answer some questions raised in paper [1] by Z. Buczolich, J.-P. Kahane, and D. Mauldin.

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This line of research was started in another joint paper with Dan Mauldin [3]. In that paper we considered a problem from 1970, originating from the Diplomarbeit (Thesis) of Heinrich von Weizsäcker [8].

Suppose that $f: (0, +\infty) \rightarrow \mathbb{R}$ is a measurable function. Is it true that $\sum_{n=1}^{\infty} f(nx)$ either converges (Lebesgue) almost everywhere or diverges almost everywhere, i.e. is there a zero-one law for $\sum f(nx)$?

This question also appeared in a paper of J. A. Haight [5]. In [5] it was proved that there exists a set $H \subset (0, \infty)$ of infinite measure, for which for all $x, y \in H$, $x \neq y$ the ratio x/y is not an integer, and furthermore

(†) for all $x > 0$, $nx \notin H$ if n is sufficiently large.

This implies that if $f(x) = \chi_H(x)$, the characteristic function of H , then $\int_0^\infty f(x) dx = \infty$ and $\sum_{n=1}^\infty f(nx) < \infty$ everywhere. Lekkerkerker in [7] started to study sets with property (†).

In [3] we answered the Haight–Weizsäcker problem.

THEOREM 1.1. *There exists a measurable function $f: (0, +\infty) \rightarrow \{0, 1\}$ and two nonempty intervals $I_F, I_\infty \subset [\frac{1}{2}, 1)$ such that for every $x \in I_\infty$ we have $\sum_{n=1}^\infty f(nx) = +\infty$ and for almost every $x \in I_F$ we have $\sum_{n=1}^\infty f(nx) < +\infty$. The function f is the characteristic function of an open set E .*

Jean-Pierre Kahane was interested in this problem and soon after our paper had become available we started to receive faxes and emails from him. This cooperation led to papers [1] and [2].

We considered a more general, additive version of the Haight–Weizsäcker problem. Since $\sum_{n=1}^\infty f(nx) = \sum_{n=1}^\infty f(e^{\log x + \log n})$, that is using the function $h = f \circ \exp$ defined on $\mathbb{R}$ and $\Lambda = \{\log n : n = 1, 2, \dots\}$ we were interested in almost everywhere convergence questions of the series $\sum_{\lambda \in \Lambda} h(x + \lambda)$.

Taking more general sets than $\Lambda = \{\log n : n = 1, 2, \dots\}$ was also motivated by a paper of Haight [6]. He proved, using the original multiplicative notation of our problem, that if $\Lambda \subset [0, +\infty)$ is an arbitrary countable set such that its only accumulation point is $+\infty$ then there exists a measurable set $E \subset (0, +\infty)$ of infinite measure such that for all $x, y \in E$, $x \neq y$, $x/y \notin \Lambda$, and for a fixed x there exist only finitely many $\lambda \in \Lambda$ for which $\lambda x \in E$. This implies that choosing $f = \chi_E$ we have $\sum_{\lambda \in \Lambda} f(\lambda x) < \infty$, but $\int_{\mathbb{R}^+} f(x) dx = \infty$.

Next we recall from [1] the definition of type 1 and type 2 sets. Given Λ an unbounded, infinite discrete set of nonnegative numbers, and a measurable $f: \mathbb{R} \rightarrow [0, +\infty)$, we consider the sum

$$s(x) = \sum_{\lambda \in \Lambda} f(x + \lambda),$$

and the complementary subsets of $\mathbb{R}$:

$$C = C(f, \Lambda) = \{x : s(x) < \infty\}, \quad D = D(f, \Lambda) = \{x : s(x) = \infty\}.$$

DEFINITION 1.2. The set Λ is of type 1 if, for every f , either $C(f, \Lambda) = \mathbb{R}$ a.e. or $C(f, \Lambda) = \emptyset$ a.e. (or equivalently $D(f, \Lambda) = \emptyset$ a.e. or $D(f, \Lambda) = \mathbb{R}$ a.e.). Otherwise, Λ has type 2.

That is for type 1 sets we have a “zero-one” law for the almost everywhere convergence properties of the series $\sum_{\lambda \in \Lambda} f(x + \lambda)$, while for type 2 sets the situation is more complicated.

DEFINITION 1.3. The unbounded, infinite discrete set $\Lambda = \{\lambda_1, \lambda_2, \dots\}$, $\lambda_1 < \lambda_2 < \dots$ is asymptotically dense if $d_n = \lambda_n - \lambda_{n-1} \rightarrow 0$, or equivalently:

$$\forall a > 0, \quad \lim_{x \rightarrow \infty} \#(\Lambda \cap [x, x + a]) = \infty.$$

If d_n tends to zero monotone decreasingly, we speak about decreasing gap asymptotically dense sets.

If Λ is not asymptotically dense we say that it is asymptotically lacunary.

We denote the non-negative continuous functions on $\mathbb{R}$ by $C^+(\mathbb{R})$, and if, in addition these functions tend to zero in $+\infty$ they belong to $C_0^+(\mathbb{R})$.

In [1] we gave some necessary and some sufficient conditions for a set Λ being of type 2. A complete characterization of type 2 sets is still unknown. We recall here from [1] the theorem concerning the Haight–Weizsäcker problem. This contains the additive version of the result of Theorem 1.1 with some additional information.

THEOREM 1.4. *The set $\Lambda = \{\log n : n = 1, 2, \dots\}$ has type 2. Moreover, for some $f \in C_0^+(\mathbb{R})$, $C(f, \Lambda)$ has full measure on the half-line $(0, \infty)$ and $D(f, \Lambda)$ contains the half-line $(-\infty, 0)$. If for each c , $\int_c^{+\infty} e^y g(y) dy < +\infty$, then $C(g, \Lambda) = \mathbb{R}$ a.e. If $g \in C_0^+(\mathbb{R})$ and $C(g, \Lambda)$ is not of the first (Baire) category, then $C(g, \Lambda) = \mathbb{R}$ a.e. Finally, there is some $g \in C_0^+(\mathbb{R})$ such that $C(g, \Lambda) = \mathbb{R}$ a.e. and $\int_0^{+\infty} e^y g(y) dy = +\infty$.*

As Λ used in the above theorem is a decreasing gap asymptotically dense set and quite often it is much easier to construct examples with lacunary Λ s, in our paper we try to give examples with a decreasing gap asymptotically dense Λ .

One might believe that for type 2 Λ s $C(f, \Lambda)$, or $D(f, \Lambda)$ are always half-lines if they differ from $\mathbb{R}$. Indeed in [1] we obtained results in this direction. A number $t > 0$ is called a translator of Λ if $(\Lambda + t) \setminus \Lambda$ is finite. Condition (*) is said to be satisfied if $T(\Lambda)$, the countable additive semigroup of translators of Λ , is dense in $\mathbb{R}^+$. We showed that condition (*) implies that $C(f, \Lambda)$ is either $\emptyset$, $\mathbb{R}$, or a right half-line modulo sets of measure zero.

In [4] we showed that this is not always the case. For a given $\alpha \in (0, 1)$ and a sequence of natural numbers $n_1 < n_2 < \dots$ we put $\Lambda^{\alpha^k} := \bigcup_{k=1}^{\infty} \Lambda_k^{\alpha^k}$, $\Lambda_k^{\alpha^k} = \alpha^k \mathbb{Z} \cap [n_k, n_{k+1})$.

If $\alpha = \frac{1}{q}$ for some $q \in \{2, 3, \dots\}$, then a slight modification of the proof of [1, Theorem 1] shows that $\Lambda^{(\frac{1}{q})^k}$ is of type 1 and condition $(*)$ is satisfied.

If $\alpha \notin \mathbb{Q}$, then one can apply [1, Theorem 5] to show that Λ^{α^k} is of type 2.

The difficult case is when $\alpha = \frac{p}{q}$ with $(p, q) = 1$, $p, q > 1$, $p < q$. In this case we showed that $\Lambda^{(\frac{p}{q})^k}$ is of type 2. In the cases $\Lambda^{(\frac{p}{q})^k}$, $(p > 1)$ condition $(*)$ is not satisfied and we also showed in [4] that there exists a characteristic function f such that $C(f, \Lambda)$ does not equal $\emptyset$, $\mathbb{R}$, or a right half-line modulo sets of measure zero. This structure of $C(f, \Lambda)$ had not been seen before our paper [4].

From the point of view of our current paper the following question [1, Question 2] is the most relevant:

QUESTION 1.5. Given open sets G_1 and G_2 when is it possible to find Λ and f such that $C(f, \Lambda)$ contains G_1 and $D(f, \Lambda)$ contains G_2 ?

It was remarked in [1] that if the counting function of Λ , $n(x) = \#\{\Lambda \cap [0, x]\}$ satisfies a condition of the type

$$\forall \ell < 0 \quad \forall a \in \mathbb{R} \quad \limsup_{x \rightarrow \infty} \frac{n(x + \ell + a) - n(x + a)}{n(x + \ell) - n(x)} < +\infty$$

(as is the case for $\Lambda = \{\log n\}$) then either $C(f, \Lambda)$ has full measure on $\mathbb{R}$ or $C(f, \Lambda)$ does not contain any interval.

It was also mentioned in [1] that if Λ is asymptotically lacunary then it is possible to construct $f \in C_0^+(\mathbb{R})$ such that both $C(f, \Lambda)$ and $D(f, \Lambda)$ have interior points.

In this paper we give an almost complete answer to Question 1.5. In Section 2 we prove Theorem 2.1. This theorem states that there is a universal decreasing gap asymptotically dense Λ such that for any open subset $G \subset \mathbb{R}$ one can find a characteristic function f_G such that $G \subset D(f_G, \Lambda)$ and $C(f_G, \Lambda) = \mathbb{R} \setminus G$ modulo sets of measure zero. We also show that one can also select a $g_G \in C_0^+(\mathbb{R})$ with similar properties.

In Section 3 we consider the question of subintervals in $C(f, \Lambda)$ when $f \in C_0^+(\mathbb{R})$. In Theorem 3.1 we prove that there exists a universal asymptotically dense infinite discrete set Λ such that for any open set $G \subset \mathbb{R}$ one can select an $f_G \in C_0^+(\mathbb{R})$ such that $D(f_G, \Lambda) = G$. In this case there is no exceptional set of measure zero, $D(f_G, \Lambda)$ equals G exactly. On the other hand, Λ is not of decreasing gap. As Theorem 3.4 shows it is impossible to find such a universal Λ with decreasing gaps. In Theorem 3.4 we prove that if Λ is a decreasing gap asymptotically dense set, $f \in C^+(\mathbb{R})$ and x is an interior point of $C(f, \Lambda)$ then $[x, +\infty) \cap D(f, \Lambda)$ is of zero Lebesgue measure.

The example provided in Theorem 3.3 demonstrates that there is a decreasing gap asymptotically dense Λ and an $f \in C_0^+(\mathbb{R})$ such that $D(f, \Lambda)$ and $C(f, \Lambda)$ both contain interior points. Of course, as Theorem 3.4 shows the interior points of $D(f, \Lambda)$ are to the left of those of $C(f, \Lambda)$.

2. A universal decreasing gap asymptotically dense Λ set

Let μ denote the one-dimensional Lebesgue measure.

We denote by $\mathbb{N} := \{n \in \mathbb{Z} : n \geq 1\}$ the set of natural numbers. For every $A, B \subset \mathbb{R}$ we put $A + B := \{a + b : a \in A \text{ and } b \in B\}$ and $A - B := \{a - b : a \in A \text{ and } b \in B\}$.

The integer, and fractional parts of $x \in \mathbb{R}$ are denoted by $[x]$ and $\{x\}$, respectively.

THEOREM 2.1. *There is a strictly monotone increasing unbounded sequence $(\lambda_0, \lambda_1, \dots) = \Lambda$ in $\mathbb{R}$ such that $\lambda_n - \lambda_{n-1}$ tends to 0 monotone decreasingly, that is Λ is a decreasing gap asymptotically dense set, such that for every open set $G \subset \mathbb{R}$ there is a function $f_G : \mathbb{R} \rightarrow [0, +\infty)$ for which*

$$(1) \quad \mu\left(\left\{x \notin G : \sum_{n=0}^{\infty} f_G(x + \lambda_n) = \infty\right\}\right) = 0,$$

and

$$(2) \quad \sum_{n=0}^{\infty} f_G(x + \lambda_n) = \infty \quad \text{for every } x \in G,$$

moreover $f_G = \chi_{U_G}$ for a closed set $U_G \subset \mathbb{R}$. By (1) and (2) we have $D(f_G, \Lambda) \supset G$, and $C(f_G, \Lambda) = \mathbb{R} \setminus G$ modulo sets of measure zero.

One can also select a $g_G \in C_0^+(\mathbb{R})$ satisfying (1) and (2) instead of f_G .

REMARK 2.2. Observe that in the above theorem we construct a universal Λ and for this set, depending on our choice of G we can select a suitable f_G such that $D(f_G, \Lambda) = G$ modulo sets of measure zero.

PROOF. Let

$$\mathcal{I} := \{(j, k) : j \in \mathbb{N} \text{ and } k \in \mathbb{Z} \cap [0, 2j \cdot 2^j)\}$$

with the following lexicographical ordering: if $(j, k), (\tilde{j}, \tilde{k}) \in \mathcal{I}$ then

$$(j, k) <_{\mathcal{I}} (\tilde{j}, \tilde{k}) \Leftrightarrow (j < \tilde{j} \text{ or } (j = \tilde{j} \text{ and } k < \tilde{k})).$$

Given $(j, k) \in \mathcal{I}$ we define its immediate successor $(\hat{j}, \hat{k})$ the following way: let $\hat{j} := j$ and $\hat{k} := k + 1$ if $k < 2j \cdot 2^j - 1$, and let $\hat{j} := j + 1$ and $\hat{k} := 0$ if

$k = 2j \cdot 2^j - 1$. It is clear that starting with $(1, 0)$ by repeated application of taking the immediate successor we can enumerate $\mathcal{I}$ and hence we will be able to do induction on $\mathcal{I}$. We will also introduce the operation of taking the predecessor of $(j, k) \neq (1, 0)$ which will be denoted by $(\check{j}, \check{k})$ and which is defined by the property $(\hat{\check{j}}, \hat{\check{k}}) = (j, k)$.

For every $(j, k) \in \mathcal{I}$ let

$$I_{j,k} := [j - (k+1)2^{-j}, j - k2^{-j}] = [a_{I_{j,k}}, b_{I_{j,k}}].$$

In (6) a set $U_{j,k}$ will be defined such that with a properly selected Λ we have

$$(3) \quad I_{j,k} \subset U_{j,k} - \Lambda = \{x \in \mathbb{R} : \exists n \in \mathbb{N} \cup \{0\} \text{ such that } x + \lambda_n \in U_{j,k}\}$$

and

$$(4) \quad \mu(\{x \in [-j, j] : \exists \text{ infinitely many } (j^*, k^*) \in \mathcal{I} \\ \text{for which } x \in (U_{j^*, k^*} - \Lambda) \setminus I_{j^*, k^*}\}) = 0.$$

Let G be an arbitrary open subset of $\mathbb{R}$ and let

$$U_G := \bigcup \{U_{j^*, k^*} : (j^*, k^*) \in \mathcal{I} \text{ and } I_{j^*, k^*} \subset G\}.$$

Put

$$(5) \quad f_G(x) := \begin{cases} 1 & \text{if } x \in U_G, \\ 0 & \text{otherwise.} \end{cases}$$

We will prove that Λ and f_G satisfy the conditions of the theorem.

Now we define the sets $U_{j,k}$. Before doing this, we recall and introduce some notation. For every $(j, k) \in \mathcal{I}$ let

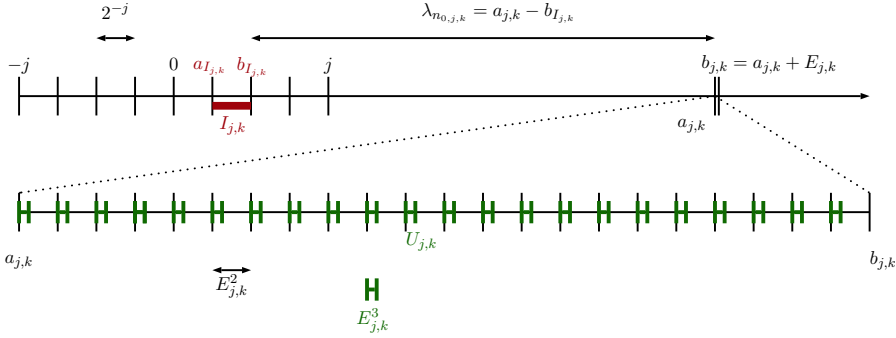
- $a_{I_{j,k}} := j - (k+1) \cdot 2^{-j}$ (that is, $a_{I_{j,k}}$ is the left endpoint of $I_{j,k}$),
- $b_{I_{j,k}} := j - k \cdot 2^{-j}$ (that is $b_{I_{j,k}}$ is the right endpoint of $I_{j,k}$),
- $E_{j,k} := 2^{-2j \cdot 2^j - k}$,
- $a_{j,k} := 2^{2j \cdot 2^j + k}$,
- $b_{j,k} := a_{j,k} + E_{j,k}$.

See Fig. 1. This and the other figure in this paper are to illustrate concepts and they are not drawn to illustrate a certain step, for example with a fixed j of our construction.

Let

$$(6) \quad U_{j,k} := \bigcup_{i=0}^{E_{j,k}^{-1}-1} [a_{j,k} + iE_{j,k}^2, a_{j,k} + iE_{j,k}^2 + E_{j,k}^3] \subset [a_{j,k}, b_{j,k}].$$

Next we prove a useful lemma.

Fig. 1: Definition of $I_{j,k}$ and $U_{j,k}$

LEMMA 2.3. For every $(j, k) \in \mathcal{I}$ we have

$$(7) \quad a_{j,k} \leq \frac{a_{\hat{j},\hat{k}}}{2} \quad \text{and} \quad E_{j,k} \geq 2E_{\hat{j},\hat{k}},$$

moreover,

$$(8) \quad E_{j,k}/2 \text{ is an integer multiple of } E_{\hat{j},\hat{k}}.$$

PROOF. It is enough to prove (7) for $a_{j,k}$ as $E_{j,k} = a_{j,k}^{-1}$.

First suppose that $k < 2j \cdot 2^j - 1$, then $\hat{j} = j$, $\hat{k} = k + 1$ and

$$(9) \quad a_{j,k} = 2^{2j \cdot 2^j + k} = \frac{2^{2j \cdot 2^j + (k+1)}}{2} = \frac{a_{\hat{j},\hat{k}}}{2}.$$

If $k = 2j \cdot 2^j - 1$ then $\hat{j} = j + 1$, $\hat{k} = 0$ and

$$(10) \quad a_{j,k} = 2^{2j \cdot 2^j + k} = 2^{2j \cdot 2^j + 2j \cdot 2^j - 1} = 2^{4j \cdot 2^j - 1} = \frac{2^{2(j+1) \cdot 2^{j+1}}}{2^{2 \cdot 2^{j+1} + 1}} = \frac{a_{\hat{j},\hat{k}}}{2^{2 \cdot 2^{j+1} + 1}}.$$

Using $E_{j,k} = a_{j,k}^{-1}$ from (9) and (10) it follows that (8) holds. $\square$

Next we turn to the definition of Λ .

During the definition of Λ we will use the notation $d_n := \lambda_n - \lambda_{n-1}$, in fact, often we will define d_n and that will provide the value of λ_n given the already defined λ_{n-1} . Let $\lambda_0 := a_{1,0} - b_{I_{1,0}}$ and $n_{0,1,0} = 0$.

Suppose that for a $(j, k) \in \mathcal{I}$ we have already defined $n_{0,j,k}$ and λ_n for $n \leq n_{0,j,k}$, $\lambda_{n_{0,j,k}} = a_{j,k} - b_{I_{j,k}}$ and $d_{n_{0,j,k}}/E_{j,k}^2$ is a positive integer (or $n_{0,j,k} = 0$). Now we need to do our next step to define these objects for $(\hat{j}, \hat{k})$.

Step $(\hat{j}, \hat{k})$. Let $n_{1,j,k} := n_{0,j,k} + 2^{-j}E_{j,k}^{-2} + 2E_{j,k}^{-1}$. For every integer $n \in [n_{0,j,k} + 1, n_{1,j,k}]$ let $d_n := E_{j,k}^2 - E_{j,k}^3$. Thus we have

$$\begin{aligned}
 (11) \quad \lambda_{n_{1,j,k}} &= \lambda_{n_{0,j,k}} + (2^{-j}E_{j,k}^{-2} + 2E_{j,k}^{-1})(E_{j,k}^2 - E_{j,k}^3) \\
 &= a_{j,k} - b_{I_{j,k}} + 2^{-j} - 2^{-j}E_{j,k} + 2E_{j,k} - 2E_{j,k}^2 \\
 &= a_{j,k} - a_{I_{j,k}} + 2E_{j,k} - 2^{-j}E_{j,k} - 2E_{j,k}^2 \\
 &= b_{j,k} - a_{I_{j,k}} + E_{j,k} - 2^{-j}E_{j,k} - 2E_{j,k}^2 \geq b_{j,k} - a_{I_{j,k}}
 \end{aligned}$$

and (from the second row of (11))

$$(12) \quad \lambda_{n_{1,j,k}} = a_{j,k} - b_{I_{j,k}} + 2^{-j} - 2^{-j}E_{j,k} + 2E_{j,k} - 2E_{j,k}^2 < a_{j,k} - b_{I_{j,k}} + 1.$$

Since $a_{j,k} - a_{I_{j,k}} = 2^{2j \cdot 2^j + k} - (j - (k + 1) \cdot 2^{-j})$ and $2^{-j}E_{j,k}$ are both integer multiples of $E_{j,k}^2 = (2^{-2j \cdot 2^j - k})^2$ from the third row of (11) we obtain that

$$(13) \quad \lambda_{n_{1,j,k}} \text{ is an integer multiple of } E_{j,k}^2.$$

By Lemma 2.3 and (12) we have

$$a_{\hat{j}, \hat{k}} - b_{I_{\hat{j}, \hat{k}}} \geq 2a_{j,k} - (j + 1) \geq a_{j,k} + j + 1 > a_{j,k} - b_{I_{j,k}} + 1 > \lambda_{n_{1,j,k}}.$$

We set

$$(14) \quad n_{0,\hat{j},\hat{k}} = n_{1,j,k} + \frac{a_{\hat{j}, \hat{k}} - b_{I_{\hat{j}, \hat{k}}} - \lambda_{n_{1,j,k}}}{2^{-1}E_{j,k}^2}$$

and

$$(15) \quad d_n = E_{j,k}^2/2 \text{ for every integer } n \in (n_{1,j,k}, n_{0,\hat{j},\hat{k}}].$$

We obtain by (14)

$$\begin{aligned}
 \lambda_{n_{0,\hat{j},\hat{k}}} &= \lambda_{n_{1,j,k}} + \frac{(n_{0,\hat{j},\hat{k}} - n_{1,j,k})E_{j,k}^2}{2} \\
 &= \lambda_{n_{1,j,k}} + a_{\hat{j}, \hat{k}} - b_{I_{\hat{j}, \hat{k}}} - \lambda_{n_{1,j,k}} = a_{\hat{j}, \hat{k}} - b_{I_{\hat{j}, \hat{k}}},
 \end{aligned}$$

and by (8), $d_{n_{0,\hat{j},\hat{k}}} = E_{j,k}^2/2$ is an integer multiple of $E_{\hat{j}, \hat{k}}^2$, hence (13) implies that

$$(16) \quad \lambda_n \text{ is an integer multiple of } E_{\hat{j}, \hat{k}}^2 \text{ for } n \in (n_{1,j,k}, n_{0,\hat{j},\hat{k}}].$$

Thus we can proceed to the next step. By repeating this procedure we can carry out the above steps for all $(j, k) \in \mathcal{I}$ and hence we can define Λ .

Now we prove (3). We fix (j, k) and choose an arbitrary point x from $I_{j,k}$. Let n_x denote the smallest integer for which

$$(17) \quad x + \lambda_{n_x} > a_{j,k}.$$

Put $n'_x := n_x + \left\lfloor \frac{x + \lambda_{n_x} - a_{j,k}}{E_{j,k}^3} \right\rfloor$.

We have $x \in I_{j,k} \subset [-j, j]$. From $x + \lambda_{n_{0,j,k}} = x + a_{j,k} - b_{I_{j,k}}$ it follows that

$$x + \lambda_{n_{0,j,k}} - a_{j,k} = x - b_{I_{j,k}} \leq 0.$$

Therefore, $n_x > n_{0,j,k}$ and hence

$$(18) \quad d_n \leq d_{n_{0,j,k}+1} = E_{j,k}^2 - E_{j,k}^3 \quad \text{for every } n \in [n_x, \infty).$$

By minimality of n_x we have

$$x + \lambda_{n_x} - a_{j,k} \leq d_{n_x} \leq E_{j,k}^2 - E_{j,k}^3.$$

Next we will show that $x + \lambda_{n'_x} \in U_{j,k}$. Using (18)

$$(19) \quad 0 \leq \left\lfloor \frac{x + \lambda_{n_x} - a_{j,k}}{E_{j,k}^3} \right\rfloor \leq \frac{d_{n_x}}{E_{j,k}^3} \leq \frac{E_{j,k}^2 - E_{j,k}^3}{E_{j,k}^3} = E_{j,k}^{-1} - 1.$$

We also infer

$$\begin{aligned} (20) \quad x + \lambda_{n'_x} &= x + \lambda_{n_x} + \sum_{n \in (n_x, n'_x]} d_n \\ &\leq x + \lambda_{n_x} + \left\lfloor \frac{x + \lambda_{n_x} - a_{j,k}}{E_{j,k}^3} \right\rfloor (E_{j,k}^2 - E_{j,k}^3) \\ &= a_{j,k} + (x + \lambda_{n_x} - a_{j,k}) + \left\lfloor \frac{x + \lambda_{n_x} - a_{j,k}}{E_{j,k}^3} \right\rfloor (E_{j,k}^2 - E_{j,k}^3) \\ &= a_{j,k} + \left\lfloor \frac{x + \lambda_{n_x} - a_{j,k}}{E_{j,k}^3} \right\rfloor E_{j,k}^2 + E_{j,k}^3 \left\{ \frac{x + \lambda_{n_x} - a_{j,k}}{E_{j,k}^3} \right\} \quad \text{using (19)} \\ &\leq a_{j,k} + (E_{j,k}^{-1} - 1)E_{j,k}^2 + E_{j,k}^3 \leq a_{j,k} + E_{j,k} = b_{j,k}. \end{aligned}$$

From (11) and (20) we obtain

$$\lambda_{n'_x} \leq b_{j,k} - x \leq b_{j,k} - a_{I_{j,k}} \leq \lambda_{n_{1,j,k}},$$

hence $n_x, n'_x \leq n_{1,j,k}$, which means that $d_n = E_{j,k}^2 - E_{j,k}^3$ for every $n \in (n_x, n'_x]$. This implies that the first inequality in (20) is, in fact an equality, that is

$$(21) \quad x + \lambda_{n'_x} = a_{j,k} + \left\lfloor \frac{x + \lambda_{n_x} - a_{j,k}}{E_{j,k}^3} \right\rfloor E_{j,k}^2 + E_{j,k}^3 \left\{ \frac{x + \lambda_{n_x} - a_{j,k}}{E_{j,k}^3} \right\}.$$

Using (19) and (21) we can see that there exists an integer $i = \left\lfloor \frac{x + \lambda_{n_x} - a_{j,k}}{E_{j,k}^3} \right\rfloor \in [0, E_{j,k}^{-1} - 1]$ such that

$$a_{j,k} + iE_{j,k}^2 \leq x + \lambda_{n'_x} \leq a_{j,k} + iE_{j,k}^2 + E_{j,k}^3,$$

that is $x + \lambda_{n'_x} \in U_{j,k}$, which implies (3).

We continue with the proof of (4). Suppose $(\check{j}, \check{k}), (j, k), (\hat{j}, \hat{k}) \in \mathcal{I}$. Then they are strictly monotone increasing in this order and are adjacent in the lexicographical ordering of $\mathcal{I}$. We have by Lemma 2.3 and the third row of (11)

$$(22) \quad \begin{aligned} j + \lambda_{n_{1,\check{j},\check{k}}} &= j + a_{\check{j},\check{k}} - a_{I_{\check{j},\check{k}}} + 2E_{\check{j},\check{k}} - 2^{-\check{j}}E_{\check{j},\check{k}} - 2E_{\check{j},\check{k}}^2 \\ &< a_{\check{j},\check{k}} + 2j + 1 \leq 2a_{\check{j},\check{k}} \leq a_{j,k}, \end{aligned}$$

that is $U_{j,k} - \lambda_{n_{1,\check{j},\check{k}}}$ is to the right of j . By (16), $\lambda_n/E_{j,k}^2$ is an integer for every $n \in (n_{1,\check{j},\check{k}}, n_{0,j,k}]$. Therefore, (22) implies that

$$(23) \quad \begin{aligned} B_{j,k} &:= [b_{j,k} - \lambda_{n_{0,j,k}}, j] \cap (U_{j,k} - \Lambda) \\ &= [b_{j,k} - \lambda_{n_{0,j,k}}, j] \cap (U_{j,k} - \{\lambda_n : n \in (n_{1,\check{j},\check{k}}, n_{0,j,k}]\}) \\ &\subset [b_{j,k} - \lambda_{n_{0,j,k}}, j] \cap \bigcup_{i \in \mathbb{Z}} [iE_{j,k}^2, iE_{j,k}^2 + E_{j,k}^3]. \end{aligned}$$

Similarly, by using (7)

$$(24) \quad \begin{aligned} -j + \lambda_{n_{0,\hat{j},\hat{k}}} &= -j + a_{\hat{j},\hat{k}} - b_{I_{\hat{j},\hat{k}}} > a_{\hat{j},\hat{k}} - (2j + 1) \\ &\geq 2a_{j,k} - (2j + 1) \geq a_{j,k} + E_{j,k} = b_{j,k}, \end{aligned}$$

that is $U_{j,k} - \lambda_{n_{0,\hat{j},\hat{k}}}$ is to the left of $-j$. Since by (13) and (15) $\lambda_n/(E_{j,k}^2/2)$ is an integer for every $n \in [n_{1,j,k}, n_{0,\hat{j},\hat{k}}]$, (24) implies that

$$(25) \quad \begin{aligned} A_{j,k} &:= [-j, a_{j,k} - \lambda_{n_{1,j,k}}] \cap (U_{j,k} - \Lambda) \\ &= [-j, a_{j,k} - \lambda_{n_{1,j,k}}] \cap (U_{j,k} - \{\lambda_n : n \in [n_{1,j,k}, n_{0,\hat{j},\hat{k}}]\}) \end{aligned}$$

$$\subset [-j, a_{j,k} - \lambda_{n_{1,j,k}}] \cap \bigcup_{i \in \mathbb{Z}} [iE_{j,k}^2/2, iE_{j,k}^2/2 + E_{j,k}^3].$$

We want to estimate the following expression from above:

$$(26) \quad \begin{aligned} & \mu([-j, j] \cap (U_{j,k} - \Lambda) \setminus I_{j,k}) \\ & \leq \mu(A_{j,k} \cup [a_{j,k} - \lambda_{n_{1,j,k}}, a_{I_{j,k}}] \cup [b_{I_{j,k}}, b_{j,k} - \lambda_{n_{0,j,k}}] \cup B_{j,k}). \end{aligned}$$

By (23) and (25) we have

$$(27) \quad \begin{aligned} \mu(A_{j,k} \cup B_{j,k}) & \leq \mu\left([-j, j] \cap \left(\bigcup_{i \in \mathbb{Z}} [iE_{j,k}^2/2, iE_{j,k}^2/2 + E_{j,k}^3]\right)\right) \\ & = E_{j,k}^3 \frac{2j}{E_{j,k}^2/2} = 4j \cdot E_{j,k}, \end{aligned}$$

and using the third row of (11), we also have

$$(28) \quad \begin{aligned} & \mu([a_{j,k} - \lambda_{n_{1,j,k}}, a_{I_{j,k}}]) \\ & = a_{I_{j,k}} - (a_{j,k} - (a_{j,k} - a_{I_{j,k}} + 2E_{j,k} - 2^{-j}E_{j,k} - 2E_{j,k}^2)) \\ & = 2E_{j,k} - 2^{-j}E_{j,k} - 2E_{j,k}^2 \leq 2E_{j,k}. \end{aligned}$$

Moreover,

$$(29) \quad \mu[b_{I_{j,k}}, b_{j,k} - \lambda_{n_{0,j,k}}] = b_{j,k} - (a_{j,k} - b_{I_{j,k}}) - b_{I_{j,k}} = b_{j,k} - a_{j,k} = E_{j,k}.$$

Writing (27), (28) and (29) into (26) yields

$$\mu([-j, j] \cap (U_{j,k} - \Lambda) \setminus I_{j,k}) \leq (4j + 3) \cdot E_{j,k}.$$

Thus

$$\begin{aligned} & \sum_{(j^*, k^*) \in \mathcal{I}} \mu([-j, j] \cap (U_{j^*, k^*} - \Lambda) \setminus I_{j^*, k^*}) \\ & \leq \sum_{\substack{(j^*, k^*) \in \mathcal{I} \\ j^* < j}} \mu([-j, j] \cap (U_{j^*, k^*} - \Lambda) \setminus I_{j^*, k^*}) \\ & + \sum_{(j^*, k^*) \in \mathcal{I}} \mu([-j^*, j^*] \cap (U_{j^*, k^*} - \Lambda) \setminus I_{j^*, k^*}) \\ & \leq \sum_{\substack{(j^*, k^*) \in \mathcal{I} \\ j^* < j}} 2j + \sum_{(j^*, k^*) \in \mathcal{I}} (4j^* + 3) \cdot E_{j^*, k^*} \end{aligned}$$

$$\begin{aligned}
&\leq 2j \cdot 2j(2^{j-1} + \dots + 1) + \sum_{j^*=1}^{\infty} \sum_{k^*=0}^{2^{j^*} \cdot 2^{j^*} - 1} (4j^* + 3) E_{j^*, k^*} \\
&\leq 4j^2 \cdot 2^j + \sum_{j^*=1}^{\infty} 2j^* \cdot 2^{j^*} (4j^* + 3) 2^{-2j^* \cdot 2^{j^*}} \\
&\leq 4j^2 \cdot 2^j + \sum_{j^*=1}^{\infty} (8(j^*)^2 + 6j^*) 2^{-2j^* \cdot 2^{j^*} + j^*} < \infty,
\end{aligned}$$

which by the Borel-Cantelli lemma implies (4).

Let G be a fixed open subset of $\mathbb{R}$. If $x \in G$, then $\{(j, k) \in \mathcal{I} : x \in I_{j, k} \subset G\}$ is an infinite set, hence according to (3) and (5)

$$\sum_{n=0}^{\infty} f_G(x + \lambda_n) = \infty.$$

If $x \in \mathbb{R} \setminus G$ and $\sum_{n=0}^{\infty} f_G(x + \lambda_n) = \infty$, then $\{n \in \mathbb{N} : x + \lambda_n \in U_G\}$ is an infinite set, which implies that $\{(j^*, k^*) \in \mathcal{I} : I_{j^*, k^*} \subset G \text{ and } x \in (U_{j^*, k^*} - \Lambda)\}$ is also infinite, indeed $x + \lambda_n \in U_G$ if and only if

there exists $(j^*, k^*) \in \mathcal{I}$ such that $I_{j^*, k^*} \subset G$ and $x + \lambda_n \in U_{j^*, k^*}$

and for a finite subset $\mathcal{J} \subset \{(j^*, k^*) \in \mathcal{I}\}$ the set $\bigcup_{(j^*, k^*) \in \mathcal{J}} U_{j^*, k^*}$ is bounded and hence this set can contain only finitely many $x + \lambda_n$. Thus (4) implies (1).

Next we see how one can modify f_G to obtain a $g_G \in C_0^+(\mathbb{R})$ still satisfying (1) and (2). In [1] there is Proposition 1, which says that one can modify f_G to obtain a $g_G \in C_0^+(\mathbb{R})$ such that $C(f_G, \lambda) = C(g_G, \lambda)$ a.e. and $D(f_G, \lambda) = D(g_G, \lambda)$ a.e. Since we want to preserve (2) we cannot change $D(f_G, \lambda)$ by an arbitrary set of measure zero. Hence in the next construction a little extra care is needed.

Put

$$(30) \quad \Lambda_N = \{\lambda \in \Lambda : \lambda \leq 10N\} \quad \text{and} \quad L_N = \#\Lambda_N.$$

Observe that $U_G \cap (-\infty, 0] = \emptyset$, U_G does not contain a half-line, and $U_G \cap [0, N]$ is the union of finitely many disjoint closed intervals for any $N \in \mathbb{N}$. Choose an open $\tilde{U}_G \supset U_G$ such that it does not contain a half-line, and

$$(31) \quad \mu((\tilde{U}_G \setminus U_G) \cap [N-1, N]) < \frac{2^{-N}}{L_N} \quad \text{for any } N \in \mathbb{N}.$$

Select a continuous function $\tilde{g}_G$ such that $\tilde{g}_G(x) = f_G(x)$ for $x \in U_G$, $\tilde{g}_G(x) = 0$ if $x \notin \tilde{U}_G$ and $|\tilde{g}_G| \leq 1$. Hence $\tilde{g}_G \geq f_G$ on $\mathbb{R}$, and $D(\tilde{g}_G, \Lambda) \supset D(f_G, \Lambda) \supset G$. It is also clear that $0 \leq \tilde{g}_G - f_G \leq \chi_{\tilde{U}_G \setminus U_G} =: h_G$, and

$$\sum_{\lambda \in \Lambda} (\tilde{g}_G(x + \lambda) - f_G(x + \lambda)) \leq \sum_{\lambda \in \Lambda} h_G(x + \lambda).$$

Next we prove that

$$(32) \quad \sum_{\lambda \in \Lambda} h_G(x + \lambda) \text{ is finite almost everywhere,}$$

yielding that $C(\tilde{g}_G, \Lambda)$ equals $C(f_G, \Lambda)$ modulo a set of measure zero. Put $H_{G,K,\infty} = \{x \in [-K, K] : \sum_{\lambda \in \Lambda} h_G(x + \lambda) = \infty\}$. We will show that

$$(33) \quad \text{for any } K > 1 \text{ we have } \mu(H_{G,K,\infty}) = 0.$$

This clearly implies (32).

Observe that if $x \in H_{G,K,\infty}$, then there are infinitely many λ s such that $x + \lambda \in \tilde{U}_G \setminus U_G$, that is, $x \in ((\tilde{U}_G \setminus U_G) - \lambda) \cap [-K, K]$. Thus, by the Borel–Cantelli lemma to prove (33) it is sufficient to show that

$$(34) \quad \sum_{\lambda \in \Lambda} \mu(((\tilde{U}_G \setminus U_G) - \lambda) \cap [-K, K]) < \infty.$$

This is shown by the following estimate

$$\begin{aligned} & \sum_{\lambda \in \Lambda} \mu(((\tilde{U}_G \setminus U_G) - \lambda) \cap [-K, K]) \\ &= \sum_{\lambda \in \Lambda} \sum_{N=1}^{\infty} \mu(((\tilde{U}_G \setminus U_G) \cap [N-1, N]) - \lambda) \cap [-K, K]) \\ &= \sum_{N=1}^{\infty} \sum_{\lambda \in \Lambda} \mu(((\tilde{U}_G \setminus U_G) \cap [N-1, N]) \cap [\lambda - K, \lambda + K]) \\ &= \sum_{N=1}^K \sum_{\lambda \in \Lambda} \mu(((\tilde{U}_G \setminus U_G) \cap [N-1, N]) \cap [\lambda - K, \lambda + K]) \\ &+ \sum_{N=K+1}^{\infty} \sum_{\lambda \in \Lambda} \mu(((\tilde{U}_G \setminus U_G) \cap [N-1, N]) \cap [\lambda - K, \lambda + K]) \end{aligned}$$

(with a finite S_1)

$$= S_1 + \sum_{N=K+1}^{\infty} \sum_{\substack{\lambda \in \Lambda \\ \lambda \leq 10N}} \mu((\tilde{U}_G \setminus U_G) \cap [N-1, N]) \cap [\lambda - K, \lambda + K])$$

(now using (30) and (31))

$$\leq S_1 + \sum_{N=K+1}^{\infty} L_N \cdot \frac{2^{-N}}{L_N} < \infty.$$

So far we have shown that $\tilde{g}_G$ satisfies (1) and (2). Since $\tilde{g}_G \in C^+(\mathbb{R})$, but not in $C_0^+(\mathbb{R})$. We need to adjust it a little further. Since G is open, choose an increasing sequence of compact sets $G_K \subset G \cap [-K, K]$ such that $\bigcup_{K=1}^{\infty} G_K = G$. Put $M_0 = 0$. Choose $M_1 \in \mathbb{R}$ such that for any $x \in G_1$ we have

$$\sum_{\substack{\lambda \in \Lambda \\ M_0+10 < \lambda < M_1}} \tilde{g}_G(x + \lambda) > 1,$$

and $\tilde{g}_G(M_1 + 5) = 0$. This latter property can be satisfied since by assumption $\tilde{U}_G$ does not contain a half-line.

In general, if we already have selected M_{K-1} such that $\tilde{g}_G(M_{K-1} + 5(K-1)) = 0$ then choose $M_K \in \mathbb{R}$ such that for any $x \in G_K$ we have

$$(35) \quad \sum_{\substack{\lambda \in \Lambda \\ M_{K-1}+10K < \lambda < M_K}} \tilde{g}_G(x + \lambda) > K,$$

and $\tilde{g}_G(M_K + 5K) = 0$. For $x \leq M_1 + 5$ we put $g_G(x) = \tilde{g}_G(x)$. For $K > 1$ and $x \in (M_{K-1} + 5(K-1), M_K + 5K]$ we put $g_G(x) = \frac{1}{K} \tilde{g}_G(x)$.

It is clear that $g_G \in C_0^+(\mathbb{R})$.

Since $g_G \leq \tilde{g}_G$ we have $C(g_G, \Lambda) \supset C(\tilde{g}_G, \Lambda)$. If we can show that $G \subset D(g_G, \Lambda)$ then we are done. Suppose $x \in G$. Then there is a K_x such that $x \in G_K$ for any $K \geq K_x$. Therefore, for these K we have $x \in [-K_x, K_x] \subset [-K, K]$ and by using (35)

$$\sum_{\substack{\lambda \in \Lambda \\ M_{K-1}+6K < \lambda < M_K+4K}} g_G(x + \lambda) = \sum_{\substack{\lambda \in \Lambda \\ M_{K-1}+6K < \lambda < M_K+4K}} \frac{1}{K} \tilde{g}_G(x + \lambda) > 1,$$

for any $K \geq K_x$ and hence $x \in D(g_G, \Lambda)$. $\square$

3. Subintervals in $C(f, \Lambda)$

THEOREM 3.1. *There exists an asymptotically dense infinite discrete set Λ such that for any open set $G \subset \mathbb{R}$ one can select an $f_G \in C_0^+(\mathbb{R})$ such that $D(f, \Lambda) = G$.*

REMARK 3.2. As Theorem 3.4 shows in the above theorem we cannot assume that Λ is a decreasing gap set. On the other hand, in our claim we have $D(f, \Lambda) = G$, that is, there is no exceptional set of measure zero where we do not know what happens. This also implies that if the interior of $\mathbb{R} \setminus G$ is non-empty then $C(f, \Lambda)$ contains intervals.



Fig. 2: Definition of I_j , U_j and related sets

PROOF. Denote by $\mathcal{I}_D = \{[(k-1)/2^l, k/2^l] : k, l \in \mathbb{Z}, l \geq 0\}$ the system of dyadic intervals. It is clear that one can enumerate the elements of $\mathcal{I}_D$ in a sequence $\{I_j\}_{j=1}^\infty$ which satisfies the following properties

$$(36) \quad I_j = [a_{I_j}, b_{I_j}] = \left[\frac{k_j - 1}{2^{l_j}}, \frac{k_j}{2^{l_j}} \right] \subset [-j, j] \quad \text{and} \quad \mu(I_j) = 2^{-l_j} \geq \frac{1}{j}.$$

We denote by $\bar{I}_j$ the closed interval which is concentric with I_j but is of length three times the length of I_j .

We put

$$U_j = [a_j, b_j] = [2^j, 2^j + 2^{-2^j}]$$

and

$$\bar{U}_j = [a_j - 2^{-2^j - j - 1}, b_j + 2^{-2^j - j - 1}] = [\bar{a}_j, \bar{b}_j].$$

See Fig. 2.

We suppose that $f_j(x) = 0$ if $x \notin \bar{U}_j$, $f_j(x) = 2^{-j}$ if $x \in U_j$, the function f_j is continuous on $\mathbb{R}$ and is linear on the connected components of $\bar{U}_j \setminus U_j$. We define

$$(37) \quad \Lambda_{1,j} = \{k \cdot 2^{-2^j - j} : k \in \mathbb{Z}\} \cap [2^j - k_j 2^{-l_j}, 2^j + 2^{-2^j} - (k_j - 1) 2^{-l_j}]$$

$$= \{k \cdot 2^{-2^j-j} : k \in \mathbb{Z}\} \cap [a_j - b_{I_j}, b_j - a_{I_j}]$$

and put $\Lambda_1 = \bigcup_{j=1}^{\infty} \Lambda_{1,j}$. Observe that if $x \in I_j$ then

$$x + \min \Lambda_{1,j} \leq b_{I_j} + \min \Lambda_{1,j} = b_{I_j} + a_j - b_{I_j} = a_j$$

and

$$x + \max \Lambda_{1,j} \geq a_{I_j} + \max \Lambda_{1,j} = a_{I_j} + b_j - a_{I_j} = b_j,$$

hence

$$(38) \quad \sum_{\lambda \in \Lambda_{1,j}} f_j(x + \lambda) \geq \frac{\text{diam } U_j}{2^{-2^j-j}} 2^{-j} = \frac{2^{-2^j}}{2^{-2^j-j}} 2^{-j} = 1.$$

On the other hand, by (36)

$$\begin{aligned} \overline{U}_j - \Lambda_{1,j} &= [\min \overline{U}_j - \max \Lambda_{1,j}, \max \overline{U}_j - \min \Lambda_{1,j}] \\ &= [\overline{a}_j - b_j + a_{I_j}, \overline{b}_j - a_j + b_{I_j}] \\ &= [a_{I_j} - 2^{-2^j} - 2^{-2^j-j-1}, b_{I_j} + 2^{-2^j} + 2^{-2^j-j-1}] \\ &\subset \left[a_{I_j} - \frac{1}{j}, b_{I_j} + \frac{1}{j} \right] \subset [a_{I_j} - 2^{-l_j}, b_{I_j} + 2^{-l_j}] = \overline{I}_j \end{aligned}$$

thus

$$(39) \quad \sum_{\lambda \in \Lambda_{1,j}} f_j(x + \lambda) = 0 \quad \text{if } x \in [-j, j], \quad x \notin \overline{I}_j.$$

Suppose $G \subset \mathbb{R}$ is a given open set and put $\mathcal{J}_G = \{j : \overline{I}_j \subset G\}$. Let $f_G(x) = \sum_{j \in \mathcal{J}_G} f_j(x)$. Then f_G is continuous and non-negative on $\mathbb{R}$ and clearly $\lim_{x \rightarrow \infty} f(x) = 0$.

We claim that

$$(40) \quad \sum_{\lambda \in \Lambda_1} f_G(x + \lambda) = +\infty$$

exactly on G .

Indeed, if $x \in G$ then there are infinitely many j s such that $x \in I_j \subset \overline{I}_j \subset G$. This means that (38) holds for infinitely many $j \in \mathcal{J}_G$ and hence (40) is true when $x \in G$.

Next we need to verify that (40) does not hold for $x \notin G$. Suppose that $j_0 \geq 10$, $j_0 \in \mathcal{J}_G$, $x \notin G$ and $x \in [-j_0, j_0]$. Then $x \notin \overline{I}_{j_0}$ and by (39) we have

$$(41) \quad \sum_{\lambda \in \Lambda_{1,j_0}} f_{j_0}(x + \lambda) = 0.$$

Next assume that $j < j_0$. Then by using (36) and (37) we have

$$\begin{aligned} \max\{x + \lambda : \lambda \in \Lambda_{1,j}\} &\leq j_0 + 2^j + 2^{-2^j} - (k_j - 1)2^{-l_j} \\ &\leq j_0 + 2^j + 2^{-2^j} + j < 2j_0 + 2^{j_0-1} + 1 < 2^{j_0} - 1 < 2^{j_0} - 2^{-2^{j_0}-j_0-1} = \bar{a}_{j_0}. \end{aligned}$$

Hence,

$$(42) \quad \sum_{\lambda \in \Lambda_{1,j}} f_{j_0}(x + \lambda) = 0.$$

If $j_0 < j$ then

$$\begin{aligned} \min\{x + \lambda : \lambda \in \Lambda_{1,j}\} &\geq -j_0 + 2^j - j \\ &> 2^{j-1} - 2j - 1 + 2^{j-1} + 1 > 2^{j_0} + 1 > \bar{b}_{j_0}, \end{aligned}$$

and hence in this case we also have (42).

Therefore, from (41) and (42) it follows that

$$(43) \quad \sum_{\lambda \in \Lambda_1} f_{j_0}(x + \lambda) = 0 \quad \text{for } j_0 \in \mathcal{J}_G, \quad j_0 \geq 10, \quad |x| \leq j_0.$$

This implies

$$\sum_{\lambda \in \Lambda_1} f_G(x + \lambda) \leq \sum_{\substack{\lambda \in \Lambda_{1,j} \\ j \leq \max\{10, |x|\}}} f_j(x + \lambda) < +\infty.$$

Since Λ_1 is not asymptotically dense we need to choose an asymptotically dense Λ_2 such that

$$(44) \quad \sum_{\lambda \in \Lambda_2} \sum_{j=1}^{\infty} f_j(x + \lambda) < +\infty \quad \text{holds for any } x \in \mathbb{R}.$$

Then for any open $G \subset \mathbb{R}$

$$\sum_{\lambda \in \Lambda_2} f_G(x + \lambda) \leq \sum_{\lambda \in \Lambda_2} \sum_{j=1}^{\infty} f_j(x + \lambda) < +\infty$$

holds and if we let $\Lambda = \Lambda_1 \cup \Lambda_2$ then Λ is asymptotically dense and $D(f_G, \Lambda) = G$.

To complete the proof of this theorem we need to verify (44) for a suitable Λ_2 . For $j \geq 10$ put

$$\Lambda_{2,j} = \{k \cdot 2^{-j} : k \in \mathbb{Z}\} \cap (2^{j-1} + 2(j-1), 2^j + 2j], \quad \text{and } \Lambda_2 = \bigcup_{j=10}^{\infty} \Lambda_{2,j}.$$

Suppose $x \in [-j_0, j_0]$ and $j_0 \geq 10$. Then for $j \geq j_0$ from $x + \lambda \in \overline{U}_j$ it follows that $2^j - 1 < x + \lambda \leq j + \lambda$, and hence

$$\lambda > 2^j - j - 1 > 2^{j-1} + 2(j-1).$$

Similarly, $x + \lambda \in \overline{U}_j$ implies $2^j + 1 > x + \lambda \geq -j + \lambda$, and hence

$$\lambda < 2^j + j + 1 < 2^j + 2j.$$

Thus from $x + \lambda \in \overline{U}_j$ it follows that $\lambda \in \Lambda_{2,j}$. Since the length of $\overline{U}_j$ is less than $2 \cdot 2^{-2^j} < 2^{-j}$ there is at most one $\lambda \in \Lambda_{2,j}$ for which $f_j(x + \lambda) \neq 0$ and for this λ we have $f_j(x + \lambda) = 2^{-j}$. Put $M_x = \max\{10, |x|\}$. Then

$$\begin{aligned} \sum_{\lambda \in \Lambda_2} \sum_{j=1}^{\infty} f_j(x + \lambda) &= \sum_{\lambda \in \Lambda_2} \sum_{j=1}^{M_x} f_j(x + \lambda) + \sum_{j=M_x+1}^{\infty} \sum_{\lambda \in \Lambda_2} f_j(x + \lambda) \\ &\leq \sum_{\lambda \in \Lambda_2} \sum_{j=1}^{M_x} f_j(x + \lambda) + \sum_{j=M_x+1}^{\infty} 2^{-j} < +\infty. \quad \square \end{aligned}$$

In Theorem 2.1 we verified that for decreasing gap asymptotically dense sets $D(f, \Lambda)$ can contain an open set, while $C(f, \Lambda)$ equals the complement of this open set only almost everywhere.

The next example shows that one can define decreasing gap asymptotically dense Λ s for which one can find nonnegative continuous f s such that both $C(f, \Lambda)$ and $D(f, \Lambda)$ have interior points.

THEOREM 3.3. *There exists a decreasing gap asymptotically dense Λ and an $f \in C_0^+(\mathbb{R})$ such that $I_1 = [1, 2] \subset D(f, \Lambda)$ and $I_2 = [4, 5] \subset C(f, \Lambda)$.*

PROOF. Put $f(x) = 2^{-2^{j+1}}$ if $x \in [10j, 10j + 1]$ for a $j \in \mathbb{N}$. Set $f(x) = 0$ if $x \in \{10j - 1/4, 10j + 5/4\}$ for a $j \in \mathbb{N}$, and also put $f(x) = 0$ for $x \leq 0$. We suppose that f is linear on the intervals where we have not defined it so far. Put $\Lambda_{1,j} = \{k \cdot 2^{-2^j} : k \in \mathbb{Z}\} \cap [10j - 10, 10j - 2)$ and $\Lambda_{2,j} = \{k \cdot 2^{-2^{j+1}} : k \in \mathbb{Z}\} \cap [10j - 2, 10j)$. Let $\Lambda = \bigcup_{j=1}^{\infty} (\Lambda_{1,j} \cup \Lambda_{2,j})$. Observe that Λ is a decreasing gap asymptotically dense set.

One can see that for $x \in I_1$ we have

$$\sum_{\lambda \in \Lambda} f(x + \lambda) \geq \sum_{j=1}^{\infty} 2^{2^{j+1}} \cdot 2^{-2^{j+1}} = +\infty$$

and for $x \in I_2$

$$\sum_{\lambda \in \Lambda} f(x + \lambda) \leq \sum_{j=1}^{\infty} 2 \cdot 2^{2^j} \cdot 2^{-2^{j+1}} < +\infty.$$

It is also clear from the construction that $\lim_{x \rightarrow \infty} f(x) = 0$. $\square$

Observe that in the above construction $I_1 \subset D(f, \Lambda)$ was to the left of $I_2 \subset C(f, \Lambda)$. The next theorem shows that for decreasing gap asymptotically dense Λ s and continuous functions this situation cannot be improved. If x is an interior point of $C(f, \Lambda)$ then the half-line $[x, \infty)$ intersects $D(f, \Lambda)$ in a set of measure zero. As Theorem 3.1 shows if we do not assume that Λ is of decreasing gap then it is possible that $D(f, \Lambda)$ has a part of positive measure, even to the right of the interior points of $C(f, \Lambda)$.

THEOREM 3.4. *Let Λ be a decreasing gap and asymptotically dense set, and let $f: \mathbb{R} \rightarrow [0, +\infty)$ be continuous. Then if x is an interior point of $C(f, \Lambda)$ then*

$$(45) \quad \mu([x, +\infty) \cap D(f, \Lambda)) = 0.$$

PROOF. Proceeding towards a contradiction assume the existence of a non-degenerate closed interval $I \subset C(f, \Lambda)$. Suppose that there is a bounded subset $D_1(f, \Lambda) \subset D(f, \Lambda)$ with positive measure to the right of I . Choose an interval $J = [a_J, b_J]$ to the right of I such that

$$(46) \quad \mu(J) = \mu(I)/10 \quad \text{and} \quad \mu(J \cap D(f, \Lambda)) = \alpha > 0.$$

We put $D_1(f, \Lambda) = J \cap D(f, \Lambda)$. We suppose that $\Lambda = \{\lambda_1, \lambda_2, \dots\}$ is indexed in an increasing order. Select N such that

$$(47) \quad \lambda_n - \lambda_{n-1} < \frac{\mu(I)}{100} \quad \text{for } n \geq N.$$

We clearly have that $\sum_{i=N}^{\infty} f(x + \lambda_i)$ diverges on $D_1(f, \Lambda)$. Moreover, if $n \in \mathbb{N}$, which is to be fixed later, for large enough M we have $\sum_{i=N}^M f(x + \lambda_i) > n$ in a set $D_2(f, \Lambda) \subset D_1(f, \Lambda)$ of measure larger than $\frac{\alpha}{2}$. Hence we have

$$(48) \quad \int_{D_2(f, \Lambda)} \sum_{i=N}^M f(x + \lambda_i) dx \geq \frac{n\alpha}{2}.$$

Assume that $i \in \{N, N+1, \dots, M\}$. We choose $\gamma(i)$ such that

$$(49) \quad a_J + \lambda_i - \lambda_{\gamma(i)} \in I, \quad \text{but} \quad a_J + \lambda_i - \lambda_{\gamma(i)+1} \notin I.$$

Since a_J is to the right of I it is clear that $\lambda_{\gamma(i)} > \lambda_i$, therefore $\gamma(i) > i \geq N$ and hence (47) implies that $\gamma(i)$ is well-defined, that is (49) can be satisfied.

It is also clear that there exists $\widetilde{M}$ such that $\gamma(i) \leq \widetilde{M}$ holds for $i \in \{N, N+1, \dots, M\}$.

By (46), (47), and (49) we have

$$(50) \quad J + \lambda_i - \lambda_{\gamma(i)} \subset I \quad \text{and hence} \quad D_2(f, \Lambda) + \lambda_i - \lambda_{\gamma(i)} \subset I.$$

Next we verify that

$$(51) \quad \text{if } i' \neq i \text{ then } \gamma(i') \neq \gamma(i).$$

Indeed, we can suppose that $i' < i$, and proceeding towards a contradiction we also suppose that $\gamma(i') = \gamma(i)$. We know that $a_J + \lambda_i - \lambda_{\gamma(i)} \in I$, moreover $a_J + \lambda_{i'} - \lambda_{\gamma(i')} \in I$ holds as well. Since $\gamma(i) = \gamma(i')$ we have

$$a_J + \lambda_{i'} - \lambda_{\gamma(i')} = a_J + \lambda_i - \lambda_{\gamma(i)} - \lambda_i + \lambda_{i'} \in I.$$

Using the first half of (49) and $\lambda_{i'} \leq \lambda_{i-1} < \lambda_i$ we also obtain

$$a_J + \lambda_i - \lambda_{\gamma(i)} - \lambda_i + \lambda_{i'} \leq a_J + \lambda_i - \lambda_{\gamma(i)} - \lambda_i + \lambda_{i-1} \in I.$$

Since Λ is of decreasing gap and $\gamma(i) > i$ we have $\lambda_{\gamma(i)+1} - \lambda_{\gamma(i)} < \lambda_i - \lambda_{i-1}$, and hence

$$a_J + \lambda_i - \lambda_{\gamma(i)} - \lambda_i + \lambda_{i-1} < a_J + \lambda_i - \lambda_{\gamma(i)} - \lambda_{\gamma(i)+1} + \lambda_{\gamma(i)} \in I,$$

which contradicts (49).

By using (50) and (51) we infer

$$(52) \quad \int_{D_2(f, \Lambda)} \sum_{i=N}^M f(x + \lambda_i) dx = \sum_{i=N}^M \int_{D_2(f, \Lambda)} f(x + \lambda_i - \lambda_{\gamma(i)} + \lambda_{\gamma(i)}) dx \\ = \sum_{i=N}^M \int_{D_2(f, \Lambda) + \lambda_i - \lambda_{\gamma(i)}} f(t + \lambda_{\gamma(i)}) dt \leq \int_I \sum_{j=N}^{\widetilde{M}} f(t + \lambda_j) dt.$$

Thus by (48) we obtain

$$\int_I \sum_{i=N}^{\widetilde{M}} f(x + \lambda_i) dx \geq \frac{n\alpha}{2},$$

as the left-hand side of (52) gives an upper bound for the integral in (48).

However, $\sum_{i=N}^{\widetilde{M}} f(x + \lambda_i)$ is continuous, which yields that this integrand is at least $\frac{n\alpha}{4\mu(I)}$ in a non-degenerate closed subinterval $I_1 \subset I$. Thus we have $s(x) = \sum_{\lambda \in \Lambda} f(x + \lambda) > \frac{n\alpha}{4\mu(I)}$ in I_1 . Hence, if we choose n to be large enough, we find that $s(x) > 1$ in I_1 .

Now by applying the very same argument to I_1 instead of I , we might obtain that $s(x) > \frac{n_1\alpha}{4\mu(I_1)}$ in a non-degenerate closed subinterval $I_2 \subset I_1$. Thus if we choose n_1 to be large enough, we find that $s(x) > 2$ in I_2 . Proceeding recursively we obtain a nested sequence of closed intervals $I_1, I_2, \dots$ such that $s(x) > k$ for $x \in I_k$. As this system of intervals has a nonempty intersection, we find that there is a point in I with $s(x) = \infty$, a contradiction. $\square$

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