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*Highlights

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> We study the stability of a pivoting cylindrical container that is slowly filled with fluid.> The stability of the equilibria are studied using the Lagrange-Dirichlet theorem. > Increasing the fluid height changes the stability of the equilibria.

Potential well metamorphosis of a pivoting fluid-filled container

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Abstract

This paper investigates the stability of a pivoting cylindrical container that is slowly filled with fluid. The action of filling the container with fluid causes the system's potential energy to evolve and modify the stability of equilibria. We analyze the stability behavior of this system and find distinct regions where edge and spill conditions require alternative expressions for the system's potential energy. The stability of the upright and tilt angle equilibria are studied using the Lagrange-Dirichlet theorem. We provide exact expressions for the potential energy of the system and bifurcation diagrams that compactly represent the stability behavior of the upright equilibria and additionally predict the presence of non-trivial or tilted equilibria. Theoretical investigations are then compared with a series of experimental tests that validate the container upright and tilted equilibria stability.

keywords

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1 Introduction

Human sensory receptors provide a motivating example where nature has used the equilibrium of a fluid to perceive balance. As noted in reference [1], fluid filled cavities enable humans to detect their cranial position and to interpret acceleration. In Meteorology, a tipping bucket type of rain sensor is commonly used as a rain detector in electronic weather stations [2]. In this apparatus, rain collects into a self-emptying bucket which destabilizes and empties once the quantity of the collected water reaches a calibrated threshold; the number of times the bucket empties and resets is then counted and used to ascertain the

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amount of rainfall. Gardens are another area where a pivoting fluid-filled container appears. In addition to the spilling bucket fountains that are sometimes used for aesthetic or watering purposes, the Japanese culture has used a fountain called the Shishi Odoshi to scare away deer for centuries [3, 4]. The present study of a fluid-filled cylinder also closely replicates the grasping of a drinking container with the human hand, where the thumb and forefinger grasp the container sides to create a pivot axis.

Still yet another application area for fluid-filled containers lies in the entertainment industry. For example, many theme parks use a self-emptying container to periodically dump large sums of water on willing participants at water parks - such as the “Tipping Tiki” in the Splash Lagoon Indoor Water Park, US. More complex attractions have also used the transition from a stable to unstable equilibria to drive devices - such as the Germaul water automat at Helburn Palace, Austria - where the lower jaw, when it is filled with water, tips over and grabs a bent rod that actuates the tongue and the eyelids. Once emptied, the lower jaw returns to its closed position and the cycle is repeated as the water is poured into the lower jaw [5, 6]. The primary connection amongst the above examples is the interaction of a fluid and rigid structure to bring about transitions in stability. Another example related to the interaction of a fluid and rigid structure is the motion control of an open container with slosh. The sloshing is a phenomenon that describes the fluid motion in partially filled containers. This phenomenon can cause the fluid to spill out of the container. Therefore controlling the motion of open containers in order to minimize the sloshing is of great interest - especially in the packaging industry [7, 8]. Also various works have appeared on the stability of containers that are immersed in a fluid. For example, Douglas studied the stability of a floating cylinder and found that for certain values of the length-diameter ratio, the cylinder attains stable floatation with its axis at one particular inclination between the horizontal and vertical [9]. Delbourgo [10] investigated the stability of different configurations of a floating plank with rectangular cross section. In another work Erdős and Schilber [11] examined the equilibrium configuration of solid prisms of squares and equilateral triangular cross section floating in a liquid. Closer to the present paper, Trahan and Kalmár-Nagy [12] studied the case of a rectangular liquid-filled vessel. They demonstrated the co-existence of several equilibrium configurations of the vessel. The equilibrium states were found by deriving the position vector of the center of gravity of the liquid, and the potential function was used to determine the stability of these equilibria. A similar system to the one considered in [12] was studied by Stépán and Bianchi [13]. They investigated the stability of a block hanging on a hook via a rope using a potential function depending on various geometrical parameters - even for container geometries that are not symmetric with respect to the in-plane axes.

The present manuscript explores the general situation of a pivoting container that is incrementally filled with fluid. The container in this study has a cylindrical shape since this geometry is perhaps the most common shape for fluid-filled containers. In order to describe the stability behavior of the container, we have identified distinct regions in the parameter space where the fluid will either spill from the container or cause a discontinuous change in the systems potential energy. Equilibria and stability are then predicted by first locating the center mass of the fluid-container system and then using the Lagrange-Dirichlet

theorem. According to the Lagrange-Dirichlet theorem [14, 15], which is applicable to the behavior of conservative systems, equilibria positions and the stability of the equilibria can be assessed from the derivatives of the potential energy expression. The same approach was used in [12] to determine the stability of the equilibrium points, however the authors in [12] assumed the mass of the vessel negligible. In this work we take into account the mass of the container. Also, the expression of the liquid center mass derived in this work is independent of the container's out-of-plan geometry, while in [12] the liquid center of mass depends on the out-of-plane geometry. Therefore, we believe that the method adopted in this paper can be applied to various container shapes.

The remainder of this paper is organized as follows. The next section provides illustrations of the system of interest and delineates between two angular regions which are defined by the angles the fluid makes with the container floor. Expressions for the potential energy, a different expression for each angular region, are then derived. The potential energy expressions are then used to solve for the equilibria and the stability of the equilibria in Section 3. This is followed by a series of bifurcation diagrams which examine the stability behavior for several interesting cases. Section 5 compares the results from a series of experimental trials with the analytical predictions. Finally, the last section provides our conclusions from our combined theoretical and experimental investigations.

2 Container potential energy domains

This section develops the potential energy expressions for a cylindrical container that is filled with an inviscid fluid, see Fig. 1(a). The liquid level is denoted by f , the container height by b , the interior container diameter by d and the distance between the pivot point and the container floor is given by p . The goal is to determine the general relationships that describe the container equilibria positions and their stability as a function of the amount of the liquid in the container.

To obtain a potential energy expression, it is important to delineate between different tilt angle regions. Specifically, Fig. 1(b,c,d) illustrate the fact that a distinction must be drawn between a tilt angle, defined by ϕ , smaller and larger than the edge angle, ϕ_e . The justification for multiple regions can be explained by recognizing that the fluid will conform to the container interior dimensions. For the case of a monotonically increasing tilt angle, the fluid will either spill or conform to the container interior dimensions. Assuming that the container does not spill, it can be observed that the fluid shape before and after the edge angle is quite different. Thus the complication in writing a potential energy expression becomes accounting for the changes within each angular region.

An expression for the edge angle, defined as the angular position where the fluid's top surface first contacts a location where container wall and floor intersect, see Fig. 1(c), is given by

$$\phi_e = \pm \arctan\left(\frac{2f}{d}\right). \quad (1)$$

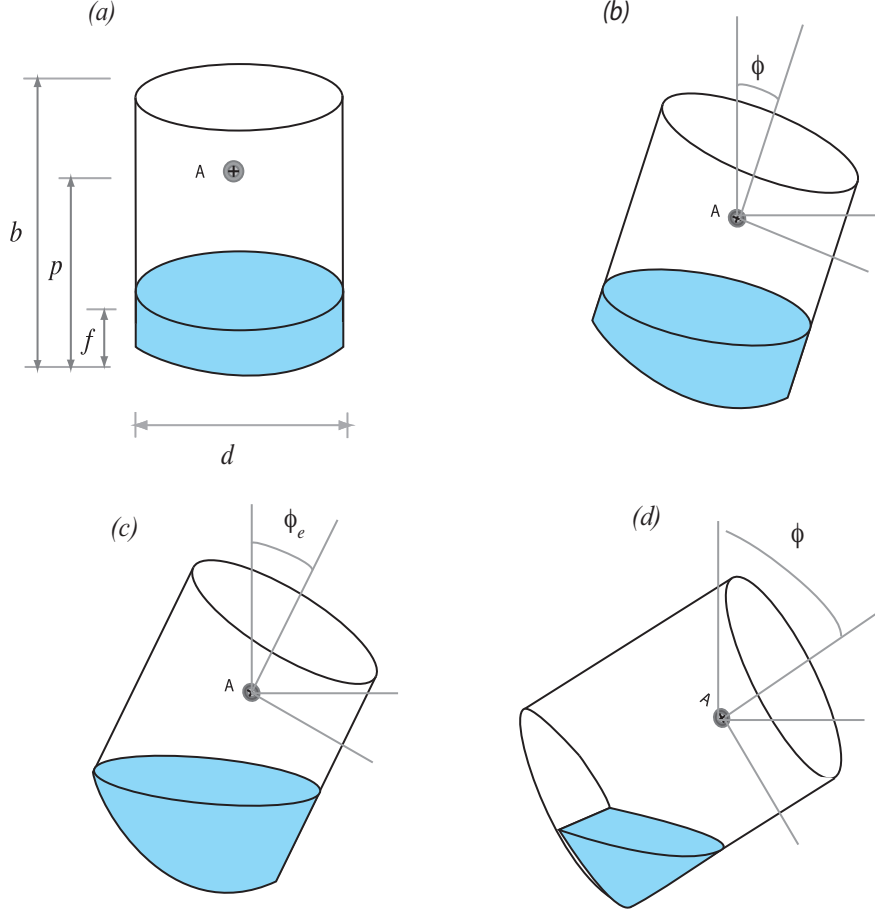


Figure 1: Schematic diagrams of a cylindrical container filled with liquid. Graphs show: (a) the upright case $\phi = 0$, (b) a tilted orientation where the fluid contacts the container wall $\phi < \phi_e$, (c) the tilted orientation where the fluid top surface exactly meets the container edge $\phi = \phi_e$, and (d) a tilted orientation where the tilt angle exceeds the edge angle $\phi > \phi_e$.

This expression can be derived from the container geometry by noting that the fluid, shown in Fig. 1(b), can be separated into a tapered region and a cylinder of height $f - d/2 \tan \phi$, as shown in Fig. 2. When the tilt angle is increased to the extent of Fig. 1c, where $\phi = \phi_e$, the height of the cylindrical fluid section must go to zero $f - d/2 \tan \phi_e = 0$. Rearranging this expression provides the result of Eq. (1).

Other important quantities are the angular positions where the fluid will spill. We call these orientations the spill angles. Indeed there are two such angles one for $\phi < \phi_e$, see Fig. 3(a), denoted by ϕ_{s1} , and the other one for $\phi > \phi_e$ and it is denoted by ϕ_{s2} , see Fig. 3(b). These two spill angles are expressed as

$$\phi_{s1} = \pm \arctan \left(\frac{2(b-f)}{d} \right), \quad (2)$$

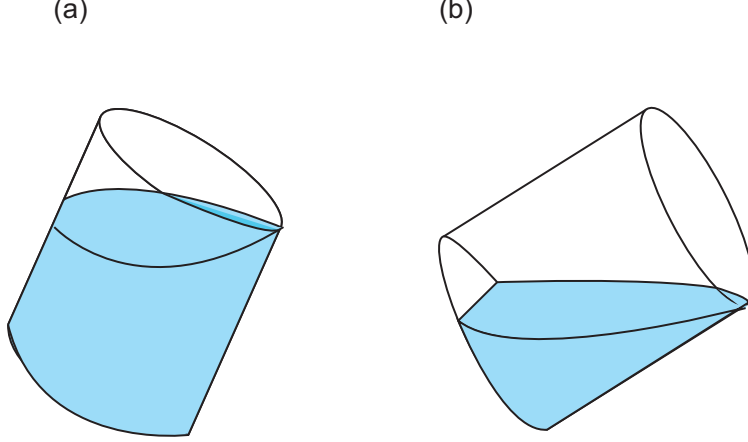


Figure 3: Schematic diagrams of the container at the two spill angles: (a) ϕ_{s1} , (b) ϕ_{s2} .

$$U(\phi) = m_f g (-\bar{y}_f \sin \phi + \bar{z}_f \cos \phi) + m_c g (-\bar{y}_c \sin \phi + \bar{z}_c \cos \phi) \quad (4)$$

where m_f is the mass of the fluid and m_c is the mass of the container. Here, the variables $(\bar{y}_f, \bar{z}_f)$ and $(\bar{y}_c, \bar{z}_c)$ are the center mass components for the fluid and the container in the rotating reference frame of Fig. 2, respectively. In comparison to the container center of mass, the relocation of the liquid center of mass is more complicated since it changes with the container tilt angle. For this purpose, following reference (Thomas & Finney 1996), the following integrals can be used to obtain the liquid center of mass as a function of the container tilt angle:

$$\bar{x}_f(\phi) = \frac{1}{m_f} \int \rho x dV, \quad (5)$$

$$\bar{y}_f(\phi) = \frac{1}{m_f} \int \rho y dV, \quad (6)$$

$$\bar{z}_f(\phi) = \frac{1}{m_f} \int \rho z dV, \quad (7)$$

where ρ is the mass density of the fluid, dV is the differential volume. Obtaining a quantitative result for the fluid center of mass as function of the container tilt angle requires the limits of integration for Eqs. (5) and (7). The equation for the plane that defines the liquid top surface $z(y, \phi)$ is the most challenging integration limit to obtain. In the paragraphs that follow, we describe one approach for finding this equation.

Instead of deriving an equation that is only applicable to a cylindrical container, a general expression is derived for an arbitrary container geometry. The procedure starts by prescribing a point in the rotating reference frame (x_o, y_o, z_o) , see Fig. 2, defined to lie in the plane of the liquid surface. Thus, the vector in the plane of the liquid top surface can be written as $\mathbf{r}_1 = (x - x_o, y - y_o, z - z_o)$. The expression for the plane can be found by taking the

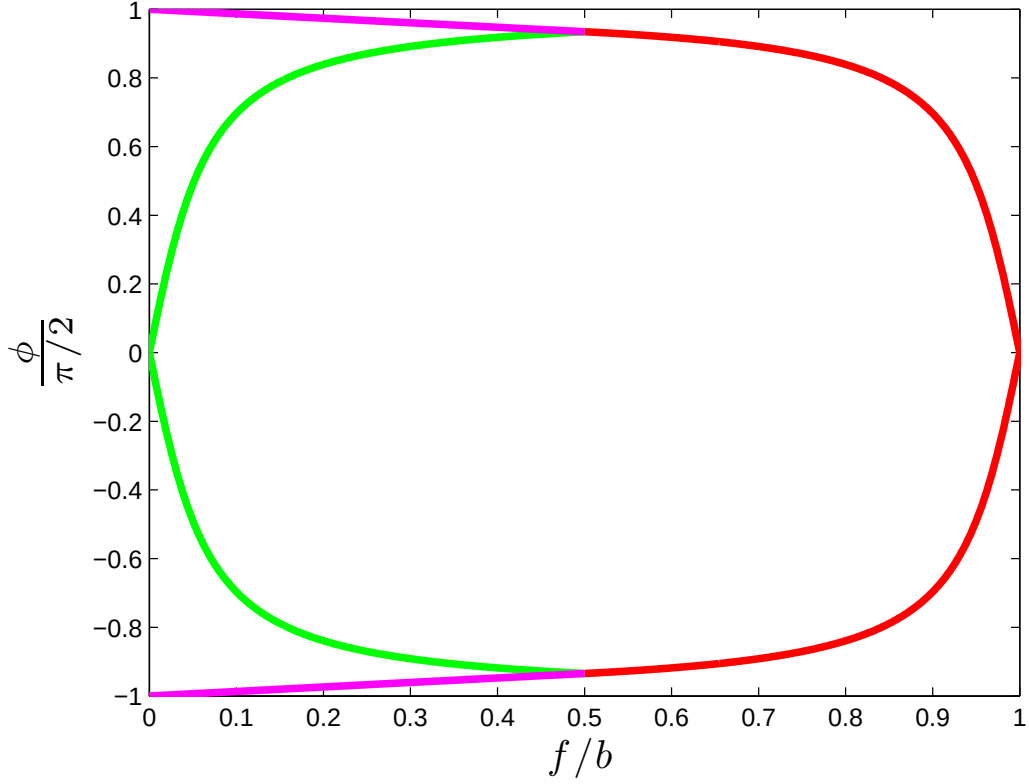


Figure 4: Plot of the edge condition angle ϕ_e (green), equation (1), and spill condition angles ϕ_{s_1} (red), Eq. (2), and ϕ_{s_2} (magenta), Eq. (2), for different f/b ratios. The fill height where $\phi_{s_i} = \phi_e$ is shown to be at $f = b/2$ and independent of the container diameter. Presented results are for $b = .92$ and $d = .095$.

dot product of a normal vector, which is normal to the plane of the liquid surface, with the vector that lies in plane of the surface

$$\mathbf{N} \cdot \mathbf{r}_1 = 0, \quad (8)$$

where $\mathbf{N}$ is the normal vector. Rather than generating a normal vector by the cross product of two vectors that lie in the plane of the liquid top surface, which requires a total of three points to be known, a clear alternative is to use the vertical unit vector $\hat{K}$ of the fixed reference frame. The only adjustment is that this vector must be transformed into the rotating coordinate frame $\mathbf{N} = (0, -\sin \phi, \cos \phi)$. Equating the dot product $\mathbf{N} \cdot \mathbf{r}_1 = 0$ results in a general expression for the plane that describes the liquid top surface

$$z(y, \phi) = (y - y_o) \tan \phi + z_o. \quad (9)$$

Eqs.(5)-(9) are independent of the container's out-of-plane geometry.

For the case of a cylindrical container, it is convenient to express the integral relationships of Eqs. (5) and (7) in polar form. The plane describing the liquid's top surface can also be written in polar form

$$z(r, \theta, \phi) = (r \sin \theta - r_o \sin \theta_o) \tan \phi + z_o, \quad (10)$$

where the notation (r_o, θ_o, z_o) is used to describe a point on the liquid top surface. As a matter of convenience, the following point along the container wall is chosen $(r_o, \theta_o, z_o) = (d/2, 3\pi/2, f - d/2 \tan \phi - p)$, the red dot in Fig. 2. After substituting $m = \rho f \pi d^2/4$, the integral relationships for the fluid center of mass in the rotating reference frame become

$$\bar{x}_f(\phi) = \frac{4}{\pi f d^2} \int_0^{2\pi} \int_0^{d/2} \int_{-p}^{z(r, \theta, \phi)} r^2 \cos \theta \, dz dr d\theta = 0, \quad (11)$$

$$\bar{y}_f(\phi) = \frac{4}{\pi f d^2} \int_0^{2\pi} \int_0^{d/2} \int_{-p}^{z(r, \theta, \phi)} r^2 \sin \theta \, dz dr d\theta = \frac{d^2}{16f} \tan \phi, \quad (12)$$

$$\bar{z}_f(\phi) = \frac{4}{\pi f d^2} \int_0^{2\pi} \int_0^{d/2} \int_{-p}^{z(r, \theta, \phi)} r z \, dz dr d\theta = \frac{f}{2} - p + \frac{(d \tan \phi)^2}{32f}. \quad (13)$$

Inserting Eqs. (12) and (13) into Eq. (4) results in an expression for the potential energy of the system

$$U(\phi) = m_f g \left[-\frac{d^2}{32f} \tan \phi \sin \phi + \left(\frac{f}{2} - p \right) \cos \phi \right] + m_c g [-\bar{y}_c \sin \phi + \bar{z}_c \cos \phi]. \quad (14)$$

The parameters $(\bar{y}_c, \bar{z}_c)$, which depends on the geometry of the container, will be given later for a specific geometry of the container. Accordingly, we note that this expression is only valid when the tilt angle is less than both the edge ϕ_e and the spill angle ϕ_{s1} , which are given by Eqs. (1) and (2), respectively.

2.2 Container potential energy when $\phi_e \leq \phi < \phi_{s_2}$

The potential energy expression of the prior section is invalid when the tilt angle exceeds the edge condition $\phi > \phi_e$. Furthermore, the results of Fig. 4 show the fluid will spill prior to the edge angle if the fluid level exceeds half the container height $f \geq b/2$. Thus a potential energy expression is required for the region $\phi > \phi_e$ and $f < b/2$.

The development of a potential energy expression requires finding the fluid center of mass as a function of the container tilt. As in the previous section, we define a point on the liquid surface before using Eq. (10) for the plane that contains the liquid top surface. A point on the line that intersects the container wall and the plane of the liquid top surface is chosen for this purpose. It is defined by $(r_o, \theta_o, z_o) = (0, \pi/2, \sqrt{2fd \tan \phi} - p)$, the red dot in Fig. 5.

Once again, the relationships for the fluid center of mass can be expressed in integral form. For instance, the relationship for $\bar{x}_f$ is

$$\bar{x}_f(\phi) = \frac{4}{\pi f d^2} \int_{-\theta_1}^{\pi+\theta_1} \int_0^{d/2} \int_{-p}^{z(r,\theta,\phi)} r^2 \cos \theta \, dz dr d\theta = 0, \quad (15)$$

where

$$\theta_1 = \arcsin \left(\sqrt{\frac{8f}{d \tan \phi}} - 1 \right). \quad (16)$$

The expressions of the fluid center of mass components $(\bar{y}_f, \bar{z}_f)$ are listed in the Appendix.

Inserting the expressions of $\bar{y}_f$ and $\bar{z}_f$, given in the Appendix, into the potential energy, Eq. (4), gives

$$\begin{aligned} U(\phi) = & m_f g \left[\frac{d^2}{128\pi f} (\cos(2\theta_1 - \phi) - \cos(2\theta_1 + \phi)) \tan \phi + \frac{dp}{2\pi f} (\sin(\theta_1 - \phi) - \sin(\theta_1 + \phi)) \right. \\ & - \frac{\sqrt{2}}{2f} \left(\frac{\theta_1}{\pi} + \frac{1}{2} \right) (d \sin \phi + 2p \cos \phi) \sqrt{df \tan \phi} + \left(\frac{\theta_1}{\pi} + \frac{1}{2} \right) \left(\frac{3d^2}{32f} \tan \phi \sin \phi + d \left(\frac{p}{2f} + 1 \right) \sin \phi \right) \\ & \left. + m_c g [-\bar{y}_c \sin \phi + \bar{z}_c \cos \phi] \right], \quad (17) \end{aligned}$$

where θ_1 is given by Eq. (16). It should be noted that the potential energy of Eq. (17) is only valid when the tilt angle is greater than the edge angle, given by Eq. (1), and less than the spill angle, (3).

Fig. 6 shows the potential energy for the two angular regions and for the case when the pivot point is above the center mass of the container, Fig. 6(a), and when it is below the center of mass, Fig. 6(b). When the pivot point is above the center mass the upright equilibrium is stable, and in the container may be viewed as a planar pendulum. While when the pivot point is moved below the center mass, the upright equilibrium becomes unstable and two stable nontrivial equilibria appear. In this figure and what follows we consider the

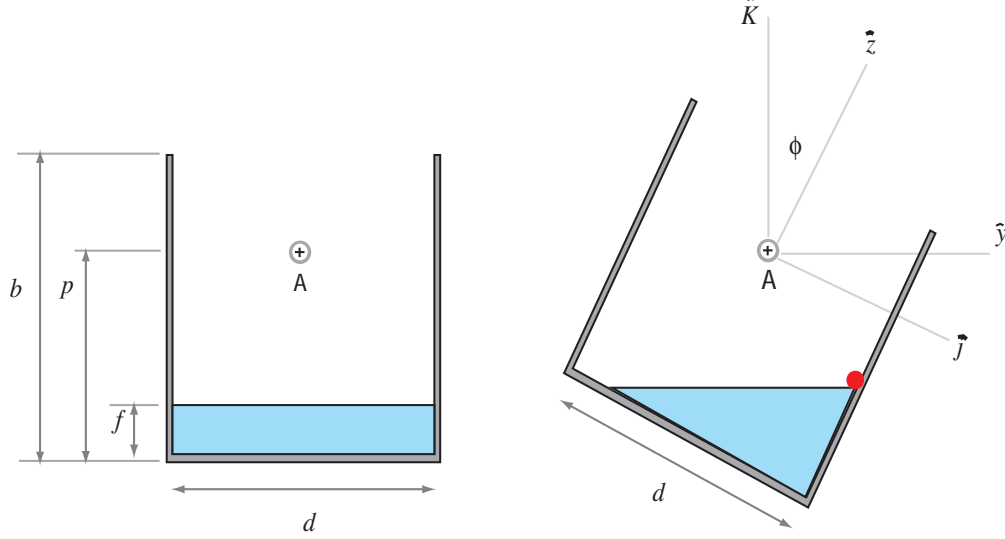


Figure 5: Schematic diagrams of the upright (left) and tilted container (right) along with associated geometry. The tilted container is shown for the case of $\phi > \phi_e$ and has a dot showing the point chosen for constructing the vector $\mathbf{r}_1$ which is used to write the potential energy when $\phi > \phi_e$.

container geometry such that $\bar{y}_c = 0$ and $z_c = -p + \eta b/2$. The parameter η is used to describe the center of mass in terms of the container half length since the cylindrical container is uncovered from the top.

In Fig. 7 the potential energy is displayed for different values of the ratio f/b . For $f/b = 0.3$, there are three equilibria a stable upright position and two unstable tilt positions, Fig. 7(a). These two tilt equilibria correspond to angles that are beyond the edge angle, Fig. 1(d). In Fig. 7(b), the tilt equilibria correspond to the edge angle, Fig. 1(c). For $f/b = 0.45$, the tilt angles are smaller than the edge angle, Fig. 1(c). As the ratio f/b is increased, $f/b = 0.6$, the tilt equilibria disappear and the upright equilibrium becomes unstable.

Figs. 7 and 8 show the potential energy for the two angular regions as defined by Eqs. (14) and (17) for two values of the parameter η .

3 Stability

This section investigates the stability of the upright and tilt angle equilibria by using the Lagrange-Dirichlet theorem. The equilibria are obtained through the first derivative of the potential energy and the stability is deduced from the second derivative. For the region corresponding to $\phi_e \leq \phi < \phi_{s1}$, the stability of the equilibria is determined numerically.

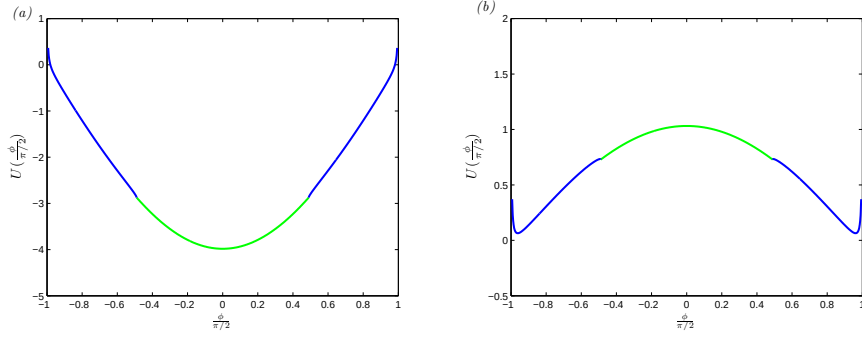


Figure 6: The potential energy for $\phi < \phi_e$ and $\phi < \phi_{s1}$ (green) Eq. (14) and the potential energy for $\phi_e < \phi < \phi_{s1}$ (blue), Eq. (17). For the parameters $g = 9.8 \text{ m/s}^2$, $\rho = 998 \text{ kg/m}^3$, $b = 0.92 \text{ m}$, $d = .095 \text{ m}$, $\eta = 0.96$, $m_c = 1.38 \text{ kg}$, $f/b = 0.05$ and (a) $p = 0.6$, (b) $p = 0.3$.

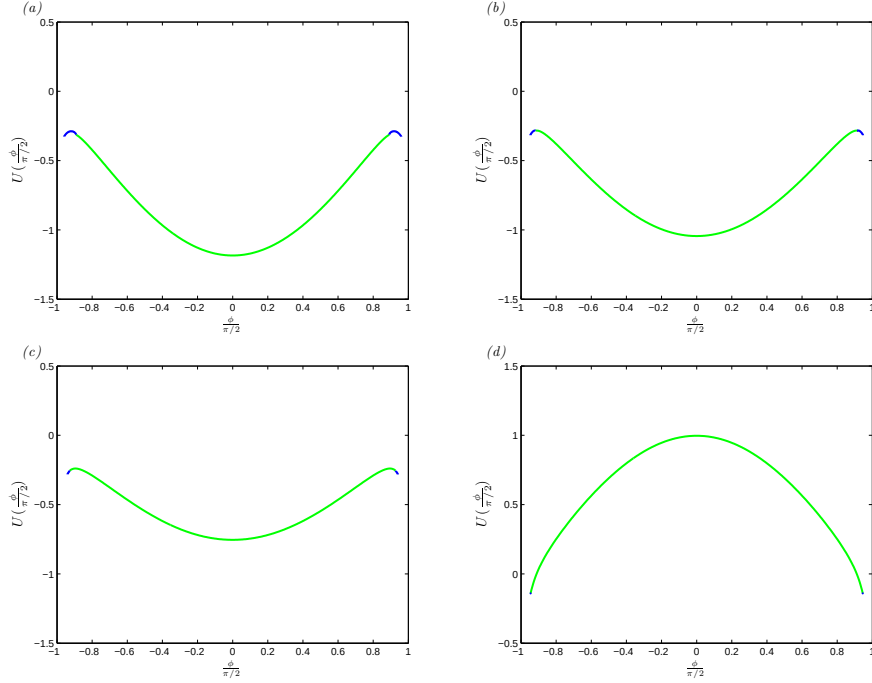


Figure 7: The potential energy for $\phi < \phi_e$ and $\phi < \phi_{s1}$ (green), Eq. (14), and the potential energy for $\phi_e < \phi < \phi_{s1}$ (blue), Eq. (17). For the parameters $g = 9.8 \text{ m/s}^2$, $\rho = 998 \text{ kg/m}^3$, $b=0.92 \text{ m}$, $d=0.095 \text{ m}$, $\eta = .96$, $p = 0.3 \text{ m}$, $m_c = 1.38 \text{ kg}$, and (a) $f/b = 0.3$, (b) $f/b = 0.4$, (c) $f/b = 0.45$, (d) $f/b = 0.6$.

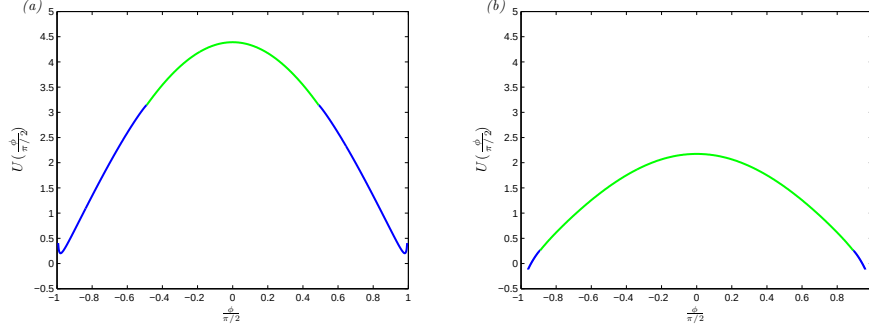


Figure 8: The potential energy for $\phi < \phi_e$ and $\phi < \phi_{s1}$ (green), Eq. (14), and the potential energy for $\phi_e < \phi < \phi_{s1}$ (blue) Eq. (17). For the parameters $g = 9.8 \text{ m/s}^2$, $\rho = 998 \text{ kg/m}^3$, $b=0.92 \text{ m}$, $d=0.095 \text{ m}$, $\eta = 1.5$, $p = 0.3 \text{ m}$, $m_c = 1.38 \text{ kg}$ and (a) $f/b = 0.05$, (b) $f/b = 0.3$.

3.1 Stability conditions of the equilibria when $\phi \leq \phi_e$ and $\phi < \phi_{s1}$

3.1.1 Equilibria

As mentioned in section 2, we use the Lagrange-Dirichlet theorem to determine equilibria and stability of the container. The equilibria are found from the first derivative of the potential energy, Eq. (14), which has the form

$$\frac{\partial U}{\partial \phi} = -\frac{\pi \rho g d^4}{128} \left[\frac{\cos^2 \phi + 1}{\cos \phi} \right] \tan \phi + g \left[\frac{\pi \rho d^2 f g}{4} \left(p - \frac{f}{2} \right) + m_c \left(p - \eta \frac{b}{2} \right) \right] \sin \phi, \quad (18)$$

where the mass of the liquid is substituted by $m = \rho f d^2 \pi / 4$ and where the coordinates of the container center of mass is substituted by $y_c = 0$ and $z_c = -p + \eta b / 2$. As mentioned before, it should be noted that the η was used to describe the center of mass in terms of the container half length. Setting Eq. (18) equals to zero and solving for ϕ , we obtain the following equilibria $\phi_u = 0$ and

$$\phi_t = \pm \arccos \left(\frac{\sqrt{\pi} d^2}{\sqrt{\frac{128 m_c}{\rho} \left(p - \eta \frac{b}{2} \right) + 32 \pi d^2 f \left(p - \frac{f}{2} \right) - \pi d^2}} \right), \quad (19)$$

where ϕ_u is the upright equilibrium of the equilibrium and ϕ_t is the tilted equilibria of the container.

3.1.2 Stability of the upright position ϕ_u

The stability of each equilibrium position is found by evaluating the second derivative of Eq. (14) at this equilibrium point. The second derivative of the potential energy is given by

$$\begin{aligned}\frac{\partial^2 U}{\partial \phi^2} &= \frac{\pi \rho g d^2}{128} \left[\frac{\cos^2 \phi - 2}{\cos^2 \phi} \right] \sin \phi \tan \phi + \frac{\pi \rho g d^2}{64} \left[16f \left(p - \frac{f}{2} \right) - \frac{d^2}{\cos^2 \phi} \right] \cos \phi \\ &+ m_c g \left(p - \frac{b}{2} \right) \cos \phi.\end{aligned}\quad (20)$$

For given parameter values of ρ , m_c , f , d , and p , the equilibria will be stable for positive values of the following expressions.

$$\frac{\partial^2 U(\phi)}{\partial \phi^2} \Big|_{\phi_u} = \frac{\pi \rho g d^2}{4} \left[f \left(p - \frac{f}{2} \right) - \frac{d^2}{16} \right] + m_c g \left(p - \eta \frac{b}{2} \right). \quad (21)$$

In the case of the upright equilibrium position $\phi_u = 0$, the fill height that defines the transition between a stable and unstable upright equilibria is found by setting Eq. (21) equal to zero:

$$f^2 - 2pf + \left[\frac{d^2}{8} - \frac{8m_c}{\pi \rho d^2} \left(p - \eta \frac{b}{2} \right) \right] = 0, \quad (22)$$

which has the following two roots

$$\begin{aligned}f_{C_1} &= p - \sqrt{p^2 - \frac{d^2}{8} + \frac{8m_c}{\pi \rho d^2} \left(p - \eta \frac{b}{2} \right)}, \\ \text{and} \\ f_{C_2} &= p + \sqrt{p^2 - \frac{d^2}{8} + \frac{8m_c}{\pi \rho d^2} \left(p - \eta \frac{b}{2} \right)}.\end{aligned}\quad (23)$$

The upright equilibrium position $\phi = \phi_u = 0$ is stable for the values of f that satisfies the following condition

$$f_{C_1} < f < f_{C_2} \quad (24)$$

3.1.3 Stability of the position ϕ_t

The stability nature of the tilt equilibrium position ϕ_t is determined by evaluating the second derivative of the potential energy, Eq. (20), at ϕ_t . Rewriting Eq. (20) and substituting ϕ_t by Eq. (19), we found that the second derivative, which is given by

$$\begin{aligned}\frac{\partial^2 U(\phi)}{\partial \phi^2} \Big|_{\phi_t} &= \sqrt{\frac{f}{32d^2} \left(p - \frac{f}{2} \right) + \frac{m_c}{8\pi \rho d^4} \left(p - \eta \frac{b}{2} \right) - \frac{1}{1024}} \\ &\quad \left[-16\pi \rho g f \left(p - \frac{f}{2} \right) - 64m_c g \left(p - \eta \frac{b}{2} \right) + \pi \rho g d^4 \right],\end{aligned}\quad (25)$$

is negative for the following values of the fluid height

$$f_{C_1} < f < f_{C_2}, \quad (26)$$

where f_{C_1} and f_{C_2} is give by Eq. (23). Therefore, the tilt equilibrium is unstable for the values of the fill heights that satisfy Eq. (26).

3.2 Stability conditions of the equilibria when $\phi_e \leq \phi < \phi_{s_1}$

In order to study the stability for the tilt angle equilibria when $\phi_e \leq \phi < \phi_{s_1}$, we numerically look for the equilibria by seeking the zeros of the first derivative of Eq. (17) and then evaluate the second derivative of Eq. (17) at these equilibria. The roots of the first derivative of the potential energy are located using Matlab *fzero* command.

Fig. 9 shows the potential energy, and the first and second derivatives of the potential energy for the two angular regions that are delineated by the two vertical blue lines, which correspond to the edge condition. This plot is produced using Eqs. (14) and (17). For simplicity we omit giving the large expressions of the first and second derivatives of Eqs. (14) and (17). Fig. 9(a) shows the existence of two stable nontrivial equilibria. As it can be seen from this figure the corresponding second derivative of these two equilibria (dashed line) is positive. The presence of non-trivial equilibrium states was also encountered in the case of a rectangular container [12]. In Fig. 9(b) these two non-trivial equilibria disappear and the upright equilibrium, which was unstable, becomes stable. In Fig. 10, the potential energy for the two angular regions is displayed. Also the stability nature of the equilibrium states are shown, with a schematic diagram of the container corresponding to each equilibrium. The vertical dashed line in the plot, Fig. 10, corresponds to the spill angle.

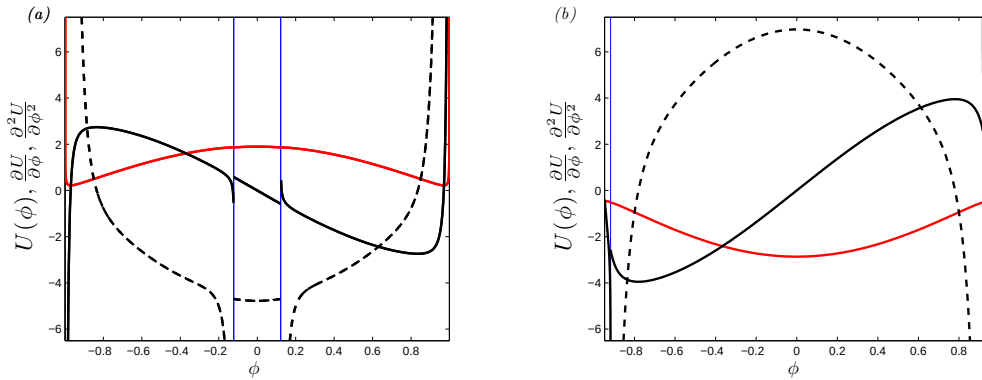


Figure 9: The potential energy (red), the first derivative (black) and the second derivative (dashed line) of the potential energy. For the parameters $g = 9.8 \text{ m/s}^2$, $\rho = 998 \text{ kg/m}^3$, $b=0.92 \text{ m}$, $d=0.095 \text{ m}$, $\eta = 1.3$, $p = 0.38 \text{ m}$, $m_c = 1 \text{ kg}$ and (a) $f/b = 0.01$, (b) $f/b = 0.4$.

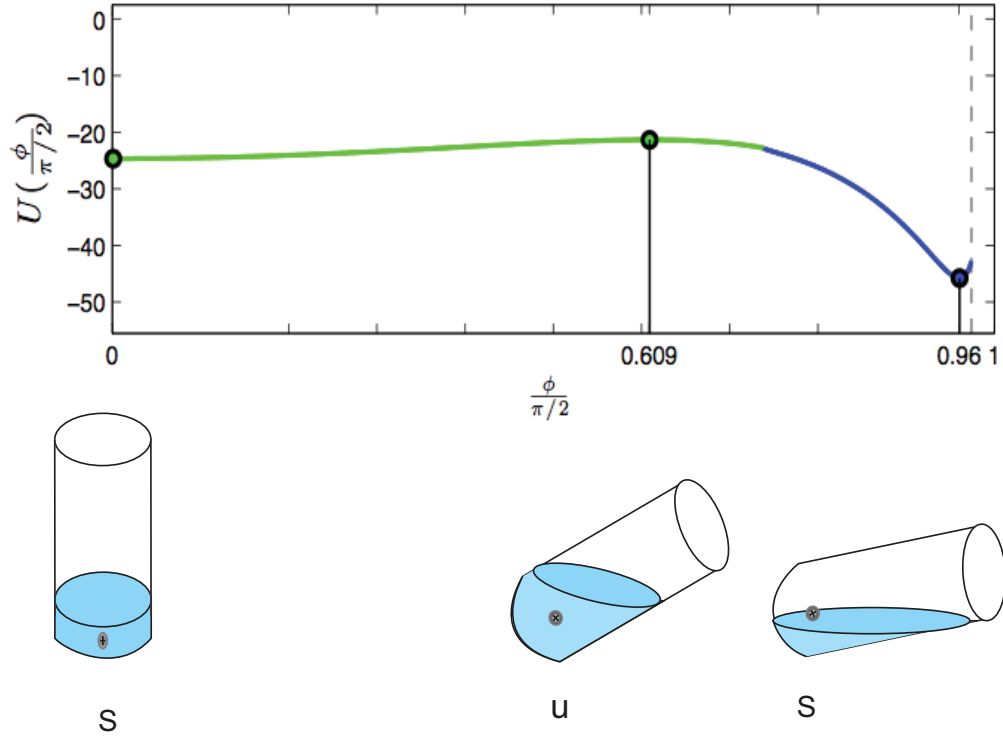


Figure 10: Potential energy for the two angular regions and three equilibrium states of the container. For the parameters $g = 9.8 \text{ m/s}^2$, $\rho = 998 \text{ kg/m}^3$, $b=3 \text{ m}$, $d=0.4 \text{ m}$, $\eta = 0.98$, $p = 0.3 \text{ m}$, $m_c = 1.3 \text{ kg}$ and $f/b = 0.46$. S: stable. U: unstable.

4 Bifurcations

Fig. 11 shows a bifurcation diagram for the container tilt as the fluid level is slowly increased. The fluid filled container undergoes a pitchfork bifurcation, see Fig. 11(a), and two successive subcritical pitchfork bifurcations at f_{C_1} and f_{C_2} , shown in Fig. 11(b,c). At the bifurcation value of the fluid height, f_{C_1} , the unstable upright equilibrium of the container becomes stable and two unstable symmetric tilt angle equilibria appear and then these two unstable tilt angle equilibria disappear again and the upright equilibrium becomes stable. The green, red and magenta lines corresponds to the edge, the first spill and the second spill, respectively. Also, from the figure we note that there are two other symmetric stable nontrivial equilibria. These stable equilibria are beyond the edge angle ϕ_e and exist near the second spill angle ϕ_{s_2} . These two stable nontrivial equilibria disappear, as the fill height is increased, either by a saddle-node bifurcation, Fig. 11b, or by reaching the second spill angle ϕ_{s_2} , (Fig. 11(c,d)). In Fig. 11(d), the upright equilibrium is unstable for all the fluid heights. This can be physically understood since $\eta = 1.2$, which means that the center of mass is located near the top of the container.

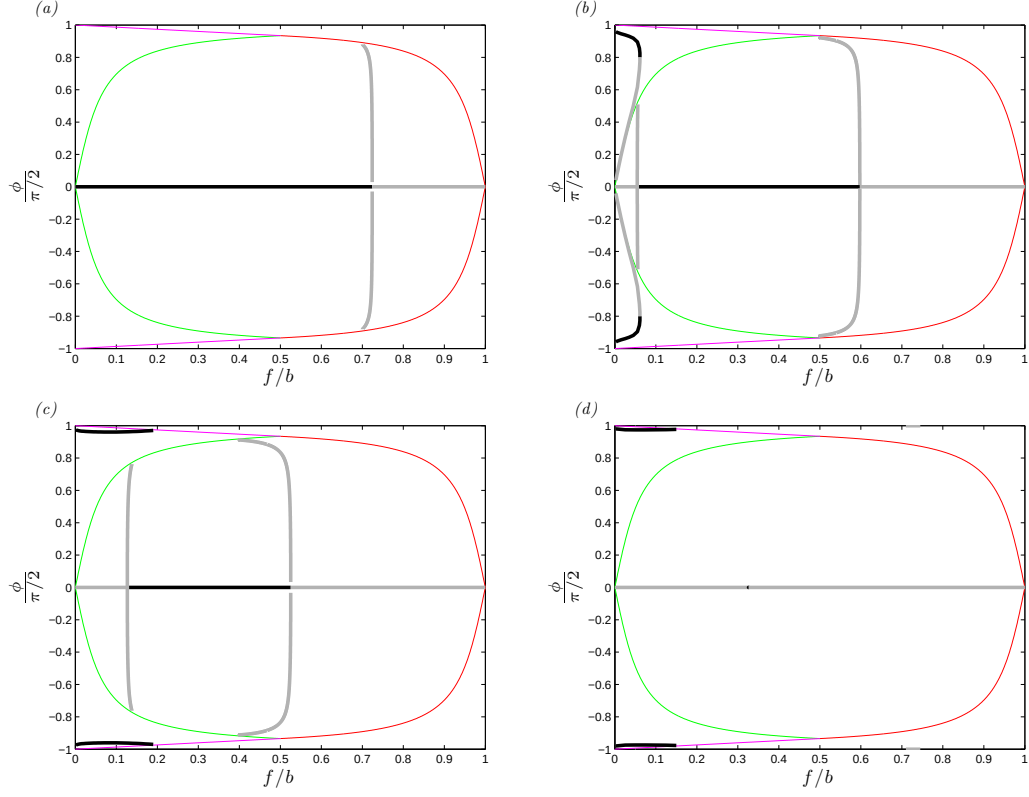


Figure 11: Bifurcation diagram of the container tilt angle as function of the fluid level. The parameters used to create these plots are: $g = 9.8 \text{ m/s}^2$, $\rho = 998 \text{ kg/m}^3$, $b = 0.92 \text{ m}$, $d = .095 \text{ m}$, $p = 0.3 \text{ m}$, $m_c = 1.38 \text{ kg}$. (a) $\eta = 0.4$. (b) $\eta = 0.8$. (c) $\eta = 0.96$. (d) $\eta = 1.2$. Stable (black). Unstable (gray).

5 Experimental validation

A picture of the experimental apparatus is shown in Fig. 12. This system was fabricated from a polycarbonate cylinder, which allowed visualization of the fluid level, and an aluminum rod and teflon inserts to minimize friction at the pivot point.

The experimental procedure consisted of incrementally filling with water. For a better visualization of the fluid height, we use a small amount of dye in the water. To check the stability of the upright equilibrium we poured water into the container until it stabilizes. Note that the upright equilibrium when the container is empty, is unstable since the pivot location is below the center of mass of the container; $p = 0.356$ m and $\eta = 0.96$. The upright equilibrium is then destabilized after the fill height reaches a certain value. The nontrivial equilibrium angle was measured with digital angle gage.

Table 1: The physical parameters of the container

Parameter	Value
b	0.92 (m)
d	0.095 (m)
m_c	1.38 (Kg)
η	0.96

Fig. 12 shows the stable upright equilibrium while Fig. 13 shows the stable tilt equilibrium that is experimentally obtained. The tilt equilibrium is beyond the edge angle as it can be seen from the figure. Fig. 14 shows the comparison between the experimental and the theoretical results. It can be seen from Fig. 14 that the theoretical results qualitatively agree with the experiment. The stable tilt equilibria lose stability via saddle-node bifurcation at certain value of the fill height. At this same value of fluid height, the upright equilibrium becomes stable after undergoing a pitchfork bifurcation. After adding more water in the container the upright equilibrium destabilizes via a second pitchfork bifurcation. The blue dots correspond to the experimental results. It should be noted that due to the imperfection of symmetry of the container and the friction, the experimental nontrivial equilibria are not symmetric. For the same reason, the upright equilibrium becomes unstable before the expected theoretical value.

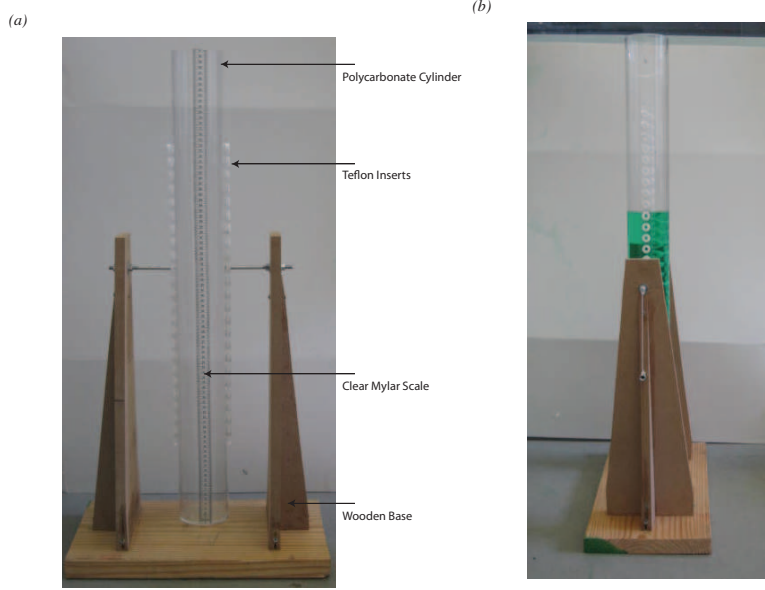


Figure 12: (a) Experimental apparatus. (b) Stable upright equilibrium of the cylindrical container for the parameters $g = 9.8 \text{ m/s}^2$, $\rho = 998 \text{ kg/m}^3$, $b = 0.92 \text{ m}$, $d = .095 \text{ m}$, $p = 0.365 \text{ m}$, $m_c = 1.38 \text{ kg}$, $\eta = 0.96$, $f = 0.516 \text{ m}$, $\phi = 0^\circ$.

6 Conclusions

This paper investigated the evolution of the stability behavior as a pivoting cylindrical container was slowly filled with fluid. Expressions were derived for the potential energy of the fluid-filled container in two angular regions - angles less than and greater than an edge condition. While these expressions accurately capture the potential energy when no liquid is lost from the container, the range of validity for the potential energy expressions must be further limited by the spill angles or the thresholds where liquid would be lost for quasi-static rotations of the container.

Theoretical investigations use the derivatives of the potential energy expressions to solve for the equilibria; in addition to the known upright equilibria, these studies unveiled the presence of non-trivial equilibria. It should be noted that the number of these non-trivial equilibria depends on the container geometry- there is a quantitative difference between our results and the ones obtained for a rectangular container [12]. Further investigations of the equilibria stability behavior have shown that some of these non-trivial equilibria can be stable. The complicated evolution of the stability behavior was then presented through the development of bifurcation diagrams which illustrate the rather complex behavior in this pedagogical system. While the intent of the present investigation was to simply explore the stability behavior of a pivoting cylindrical container, we note that various versions of the system under consideration appear in many places. More specifically, applications range from the design of a drinking container (when held by the thumb and forefinger) to explaining the design of tipping buckets that have appeared in theme parks.



Figure 13: Stable tilt equilibrium of the cylindrical container for the parameters $g = 9.8 \text{ m/s}^2$, $\rho = 998 \text{ kg/m}^3$, $b = 0.92 \text{ m}$, $d = .095 \text{ m}$, $p = 0.365 \text{ m}$, $m_c = 1.38 \text{ kg}$, $\eta = 0.96$, $f = 0.062 \text{ m}$, $\phi = 81^\circ$.

We have additionally compared our theoretical findings with a series of experimental tests. In addition to corroborating our predictions for the stability of the upright equilibria, the experimental findings confirm the possibility of stable non-trivial or tilted equilibria. Experimental findings also provide very close agreement with the theoretical predictions for the transitions in stability behavior .

Appendix

$$\begin{aligned}
\bar{y}_f(\phi) &= \frac{4}{\pi f d^2} \int_{-\theta_1}^{\pi+\theta_1} \int_0^{d/2} \int_{-p}^{z(r,\theta,\phi)} r^2 \sin \theta \, dz dr d\theta \\
&= \frac{d^2 \tan \phi}{32\pi f} \left[2 \arcsin \left(2\sqrt{2} \sqrt{\frac{f \cot \phi}{d}} - 1 \right) - \sin \left(2\sqrt{2} \sqrt{\frac{f \cot \phi}{d}} - 1 \right) \right] \\
&\quad + \frac{1}{6\pi f} \sqrt{4\sqrt{2} \sqrt{\frac{f \cot \phi}{d}} - \frac{8f \cot \phi}{d}} \left[2\sqrt{2} d \sqrt{df \tan \phi} - d^2 \tan \phi \right] + \frac{d^2 \tan \phi}{32f} \\
\bar{z}_f(\phi) &= \frac{4}{\pi f d^2} \int_{-\theta_1}^{\pi+\theta_1} \int_0^{d/2} \int_{-p}^{z(r,\theta,\phi)} r z \, dz dr d\theta \\
&= \arcsin \left(2\sqrt{2} \sqrt{\frac{f}{\tan \phi}} - 1 \right) \left[\frac{d}{\pi} \left(1 + \frac{p}{2f} \right) \tan \phi - \frac{\sqrt{2df \tan \phi}}{2\pi f} (2p + d \tan \phi) + \frac{5d^2 \tan^2 \phi}{32\pi f} \right] \\
&\quad - \frac{2d}{6\pi f} \sqrt{2} \sqrt{\sqrt{\frac{f}{d \tan \phi}} - \frac{8f}{d \tan \phi}} (2p + d \tan \phi) \tan \phi + \frac{d}{2} \left(1 + \frac{p}{2f} \right) \tan \phi
\end{aligned} \tag{27}$$

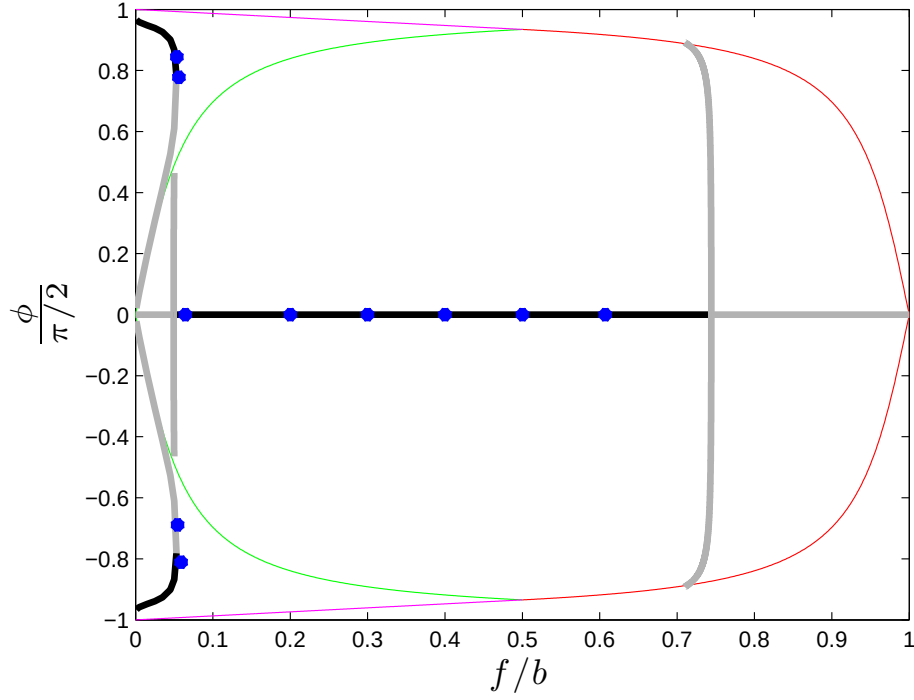


Figure 14: Bifurcation diagram of the container angle equilibria for the parameters $g = 9.8 \text{ m/s}^2$, $\rho = 998 \text{ kg/m}^3$, $b = 0.92 \text{ m}$, $d = .095 \text{ m}$, $p = 0.365 \text{ m}$, $m_c = 1.38 \text{ kg}$. $\eta = 0.96..$ Stable (black). Unstable (gray). Experimental results (blue dots).

$$- \frac{d^2 \tan^2 \phi}{64\pi f} \sin \left(2 \arcsin \left(2\sqrt{2} \sqrt{\frac{f}{d \tan \phi}} - 1 \right) \right) + \frac{5d^2 \tan^2 \phi}{64f} \quad (28)$$

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