

CURVATURE AND TORSION OF GEODESICS IN THREE HOMOGENEOUS RIEMANNIAN 3-GEOMETRIES

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ABSTRACT. In this paper we study the geodesics in three of the eight homogeneous Riemannian 3-geometries. The torsion and curvature of geodesics in *Nil*, *Sol* and $\widetilde{SL(2, \mathbb{R})}$ geometry are calculated and discussed.

1. INTRODUCTION

A few years ago G. I. Perelman gave a proof of the Thurston conjecture stating that a compact three-manifold with a given topology has a canonical decomposition into a sum of 'simple three-manifolds', whose interiors each admits one, and only one metric, of eight homogeneous Riemannian geometries: E^3 , H^3 , S^3 , $S^2 \times \mathbb{R}$, $H^2 \times \mathbb{R}$, $\widetilde{SL(2, \mathbb{R})}$, *Nil* and *Sol* [5, 6, 7].

This work is a proof of a famous special case of the Thurston conjecture, the Poincaré conjecture (a closed three-manifold with trivial fundamental group is necessarily homeomorphic to the 3-sphere).

After that, the interest in studying these eight homogeneous Riemannian 3-geometries increased. Our interest in *Nil*, *Sol* and $\widetilde{SL(2, \mathbb{R})}$ geometries is caused by the fact that the other (five) Thurston geometries have constant curvature, or are product spaces of geometries having constant curvature.

2. THE GENERAL METHOD OF COMPUTING CURVATURE AND TORSION

In [1] A. Bölcskei and B. Szilágyi showed the way of computing the curvature (κ) and torsion (τ) of an arbitrary curve (having a known parametric form) in the *Sol* geometry:

$$\kappa(t) = |\dot{\mathbf{r}}(t)|^{-3} \sqrt{\begin{vmatrix} \langle \dot{\mathbf{r}}(t), \dot{\mathbf{r}}(t) \rangle & \langle \dot{\mathbf{r}}(t), \ddot{\mathbf{r}}(t) \rangle \\ \langle \ddot{\mathbf{r}}(t), \dot{\mathbf{r}}(t) \rangle & \langle \ddot{\mathbf{r}}(t), \ddot{\mathbf{r}}(t) \rangle \end{vmatrix}}, \quad (2.1)$$

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$$\tau(t) = P \sqrt{\begin{vmatrix} \langle \dot{\mathbf{r}}(t), \dot{\mathbf{r}}(t) \rangle & \langle \dot{\mathbf{r}}(t), \ddot{\mathbf{r}}(t) \rangle & \langle \dot{\mathbf{r}}(t), \ddot{\ddot{\mathbf{r}}}(t) \rangle \\ \langle \ddot{\mathbf{r}}(t), \dot{\mathbf{r}}(t) \rangle & \langle \ddot{\mathbf{r}}(t), \ddot{\mathbf{r}}(t) \rangle & \langle \ddot{\mathbf{r}}(t), \ddot{\ddot{\mathbf{r}}}(t) \rangle \\ \langle \ddot{\ddot{\mathbf{r}}}(t), \dot{\mathbf{r}}(t) \rangle & \langle \ddot{\ddot{\mathbf{r}}}(t), \ddot{\mathbf{r}}(t) \rangle & \langle \ddot{\ddot{\mathbf{r}}}(t), \ddot{\ddot{\mathbf{r}}}(t) \rangle \end{vmatrix}} \begin{vmatrix} \langle \dot{\mathbf{r}}(t), \dot{\mathbf{r}}(t) \rangle & \langle \dot{\mathbf{r}}(t), \ddot{\mathbf{r}}(t) \rangle \\ \langle \ddot{\mathbf{r}}(t), \dot{\mathbf{r}}(t) \rangle & \langle \ddot{\mathbf{r}}(t), \ddot{\mathbf{r}}(t) \rangle \end{vmatrix}, \quad (2.2)$$

where

$$P = D \cdot |\dot{\mathbf{r}}(t)|^{-12}$$

and D denotes the next numerical determinant:

$$D = \begin{vmatrix} \dot{x}(t) & \dot{y}(t) & \dot{z}(t) \\ \ddot{x}(t) & \ddot{y}(t) & \ddot{z}(t) \\ \ddot{\ddot{x}}(t) & \ddot{\ddot{y}}(t) & \ddot{\ddot{z}}(t) \end{vmatrix}$$

and t is an arbitrary parameter.

Naturally, these formulas are effective in an arbitrary homogeneous Riemannian 3-geometry having a known metric tensor field, that determines the inner product used in formulas (2.1) and (2.2).

3. THE NIL GEOMETRY

The geodesics of *Nil* geometry originating from the famous $L(\mathbb{R})$ matrix group (studied by Werner Heisenberg) have been determined by Emil Molnár in [3]. The metric tensor field of *Nil* is given by

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 + x^2 & -x \\ 0 & -x & 1 \end{pmatrix}. \quad (3.1)$$

The parametric form of the geodesics starting in the origin is

$$\begin{aligned} x(t) &= \frac{c}{\omega} (\sin(\omega t + \alpha) - \sin \alpha), \\ y(t) &= -\frac{c}{\omega} (\cos(\omega t + \alpha) - \cos \alpha), \\ z(t) &= \omega t + \frac{c^2}{2\omega} t - \frac{c^2}{4\omega^2} (\sin(2\omega t + 2\alpha) - \sin 2\alpha) + \\ &\quad + \frac{c^2}{2\omega^2} (\sin(\omega t + 2\alpha) - \sin 2\alpha - \sin \omega t), \end{aligned} \quad (3.2)$$

where α is the polar angle and $c = \cos \theta$ and $\omega = \sin \theta$, where θ is the azimuth angle.

If $\omega = 0$ ($\theta = 0$), the geodesics have a simpler parametric form:

$$x(t) = c \cos \alpha t, \quad y(t) = c \sin \alpha t, \quad z(t) = \frac{c^2}{2} \cos \alpha \sin \alpha t^2, \quad (3.3)$$

where (naturally) $c = \cos \theta = 1$.

In a more formal way, (3.2) and (3.3) are the solutions of the well-known differential equation (system) of geodesics

$$\frac{d^2 u^k}{dt^2} + \Gamma_{ij}^k \frac{du^i}{dt} \frac{du^j}{dt} = 0 \quad (3.4)$$

(where Γ_{ij}^k ($i, j, k, l \in \{1, 2, 3\}$) are the Christoffel-symbols obtained from the metric tensor in the following way:

$$\Gamma_{ij}^k = \frac{1}{2} \left(\frac{\partial g_{jl}}{\partial u^i} + \frac{\partial g_{li}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^l} \right) g^{lk}, \quad (3.5)$$

and $u^1 = x$, $u^2 = y$, $u^3 = z$) with the following initial conditions

$$\dot{x}(0) = c \cdot \cos \alpha, \quad \dot{y}(0) = c \cdot \sin \alpha, \quad \dot{z}(0) = \omega. \quad (3.6)$$

Because of the length of the formulas, most of the computations were done by computer. Substituting (3.2) into the formulas (2.2) and (2.1) one get a large form of the torsion that can be written in a simpler form with the notation $A = \cos(2(\alpha + \omega t))$

$$\tau(t) = \left(\frac{\omega}{2}\right)^4 (\omega^2 - 1) [3(\omega^2 - 1)A - (\omega^2 + 1)]^2 [(\omega^2 + 1)^2 - A^2(\omega^2 - 1)^2]. \quad (3.7)$$

Very similarly, we obtain the form of the curvature

$$\kappa(t) = \frac{\sqrt{1 + 5\omega^2 - 5\omega^4 - \omega^6 + (\omega^2 - 1)^3 \cos(4\alpha + \omega t)}}{2\sqrt{2}}$$

that can be simplified by introducing $B = \cos(4(\alpha + \omega t))$

$$\kappa(t) = \frac{1}{2\sqrt{2}} [(1 - B)(1 - \omega)^3 - 8\omega^2(\omega^2 + 1)]. \quad (3.8)$$

In the $\theta = 0$ case we get

$$\tau \equiv 0, \quad \kappa \equiv \frac{1}{2} \sin |2\alpha|. \quad (3.9)$$

Note that κ and τ are independent from the parameter.

4. THE SOL GEOMETRY

"This (the Sol geometry) is the real weird. Unlike the previous geometries, solve geometry isn't even rotationally symmetric. I don't know any good intrinsic way to understand it." (J. R. WEEKS) [9]

The Sol geometry can be obtained by giving a group structure to the semi-direct product $\mathbb{R} \ltimes \mathbb{R}^2$ as follows:

$$(1 \quad a \quad b \quad c) \begin{pmatrix} 1 & x & y & z \\ 0 & e^{-z} & 0 & 0 \\ 0 & 0 & e^z & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = (1 \quad x + ae^{-z} \quad y + be^z \quad z + c). \quad (4.1)$$

This is the right action by a translation $(1, x, y, z)$ on a point $(1, a, b, c)$ yielding also a point of Sol expressed in homogeneous coordinates, for (x, y, z) and (a, b, c) , after choosing a fixed origin $O(1, 0, 0, 0)$.

Then an invariant metric on $Sol(O, T)$ is given by

$$(ds)^2 = (dx)^2 e^{2z} + (dy)^2 e^{-2z} + (dz)^2, \quad (4.2)$$

as arc length square, now in any point $(1, x, y, z)$ [8],[4].

Therefore the metric tensor field of Sol is given by

$$g_{ij} = \begin{pmatrix} e^{2z} & 0 & 0 \\ 0 & e^{-2z} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (4.3)$$

The parametric forms of geodesics satisfying the initial conditions

$$0 = x(0) = y(0) = z(0), \quad \dot{x}(0) = u, \quad \dot{y}(0) = v, \quad \dot{z}(0) = w, \quad u^2 + v^2 + w^2 = 1$$

are given in [1]. Following the notation of that paper, we study five essentially different cases (depending on the values of u, v and w).

1. $u \neq 0, v \neq 0, 0 < |w| = \sqrt{1 - u^2 - v^2} < 1$,
2. $u \neq 0, v \neq 0, w = 0$,
3. $v = 0, 0 < |w| = \sqrt{1 - u^2} < 1$,
4. $u = 0, 0 < |w| = \sqrt{1 - v^2} < 1$,
5. $u = 0, v = 0, |w| = 1$.

In cases 1, 2 and 5 the results of [1] have been recomputed. In cases 3 and 4 general forms of curvature and torsion are obtained.

4.1. The $v=0$ case (Nr. 3.)

The parametric form of the geodesics:

$$x(t) = u \frac{\sinh t}{\cosh t + w \sinh t}, \quad y(t) = 0, \quad z(t) = \ln(\cosh t + w \sinh t). \quad (4.4)$$

Using the general formula of curvature, we get the quantity

$$\kappa_3(t) = \sqrt{1 - \omega^2} \left[\frac{(1 + \omega^2) \cosh 2t + 2\omega \sinh 2t}{(\cosh t + \omega \sinh t)^3} \right]$$

that can be written in an equivalent (and maybe more picturesque) form:

$$\kappa_3(t) = 4\sqrt{1 - \omega^2} \left[\frac{((1 + \omega)e^t)^2 + ((1 - \omega)e^{-t})^2}{((1 + \omega)e^t + (1 - \omega)e^{-t})^3} \right]. \quad (4.5)$$

(The lower index of κ denotes the number of the case.)

This general form gives the curvature in the origin:

$$\kappa_3(0) = \sqrt{1 - w^2}(1 + \omega^2). \quad (4.6)$$

(The result of [1] has been recomputed.)

4.2. The $u=0$ case (Nr. 4.)

The parametric form of the geodesics:

$$x(t) = 0, \quad y(t) = v \frac{\sinh t}{\cosh t - w \sinh t}, \quad z(t) = -\ln(\cosh t - w \sinh t). \quad (4.7)$$

In this similar case we have a similar result

$$\kappa_4(t) = \sqrt{1 - \omega^2} \left[\frac{(1 + \omega^2) \cosh 2t - 2\omega \sinh 2t}{(\cosh t - \omega \sinh t)^3} \right].$$

The equivalent form

$$\kappa_4(t) = 4\sqrt{1 - \omega^2} \left[\frac{((1 - \omega)e^t)^2 + ((1 + \omega)e^{-t})^2}{((1 - \omega)e^t + (1 + \omega)e^{-t})^3} \right]. \quad (4.8)$$

And the curvature in the origin is the same as in case 3 (in this way the result of [1] has been checked),

$$\kappa_4(0) = \sqrt{1 - \omega^2}(1 + \omega^2). \quad (4.9)$$

Note that $\kappa_3(t, \omega) = \kappa_4(t, -\omega) = \kappa_3(-t, -\omega) = \kappa_4(-t, \omega)$, and that implies (among others), that the curvature in the origin is the same in cases 3 and 4.

$\tau \equiv 0$ has been checked in all four investigated cases.

5. THE $\widetilde{SL(2, \mathbb{R})}$ GEOMETRY

The $\widetilde{SL(2, \mathbb{R})}$ is a universal covering group of $SL(2, \mathbb{R})$, and (at the same time) a 3-dimensional Lie group of all 2×2 real matrices with determinant 1. $\widetilde{SL(2, \mathbb{R})}$ is also a Lie group and admits a metric invariant under right (or left, but not both) multiplication. The geometry of $\widetilde{SL(2, \mathbb{R})}$ arises naturally as the geometry of the hyperbolic plane $\mathbb{H}^2$ and it is similar to the Nil geometry in a sense, that it is a twisted line bundle. However, while Nil is a twisted bundle over Euclidean plane, $\widetilde{SL(2, \mathbb{R})}$ is a twisted bundle over $\mathbb{H}^2$.

In the Riemannian geometry of $\widetilde{SL(2, \mathbb{R})}$, we can define the infinitesimal arc length square with the aid of Lie algebras: the symmetric metric tensor field (g_{ij}) is given by (See:[2])

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sinh^2 r (\cosh^2 r + \sinh^2 r) & \sinh^2 r \\ 0 & \sinh^2 r & 1 \end{pmatrix}. \quad (5.1)$$

The parametric form of the geodesics starting in the origin is given in [2]. We have computed the torsion and curvature in all cases, but the forms are (still) not simple enough to discuss (in general). But a special geodesics has manageable torsion and curvature, and these quantities have a simple relation in this case. This geodesics satisfies the initial conditions

$$r(0) = 0, \quad \dot{r}(0) = \dot{\varphi}(0) = -\dot{\vartheta}(0) = \frac{\sqrt{2}}{2}. \quad (5.2)$$

(This is the "light direction"-case.)

The parametric form of the geodesics:

$$\begin{aligned} r(s) &= \operatorname{arcsinh} \left(\frac{\sqrt{2}}{2} s \right), \quad \vartheta(s) = -\arctan \left(\frac{\sqrt{2}}{2} s \right), \\ \varphi(s) &= \sqrt{2}s - \arctan \left(\frac{\sqrt{2}}{2} s \right). \end{aligned} \quad (5.3)$$

Using the formulas (2.1) and (2.2) we have the quantities

$$\kappa(t) = \frac{1}{16} \sqrt{\frac{(9+4t^2)t^2}{(1+t^2)^7}} \quad (5.4)$$

and

$$\tau(t) = \frac{t^9(9+4t^2)}{32\sqrt{2}(1+t^2)^{15/2}(2+t^2)^4}. \quad (5.5)$$

It is easy to check, that the curvature is bounded.

The torsion can be simply expressed with the curvature

$$\tau(t) = \frac{4\sqrt{2}t^7}{\sqrt{1+t^2}(2+t^2)^4} \kappa^2(t). \quad (5.6)$$

Observe, that (5.6) implies that the torsion is bounded too.

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