

SOME APPLICATION OF OPTIMAL AVERAGE IN RAYTRACING BASED VOLUME VISUALIZATION

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ABSTRACT

Volumetric visualization techniques have become popular nowadays, because the use of these methods permits to visualize the inner-structure of the data whenever it exists. The first author quite recently prepared a TDK thesis ([5]) in this area of which the present is work part. To achieve our goal we implemented error analysis as well as optimal average to choose the optimal sample to deal with. We applied these methods in the visualization of deterministic hyper-complex fractals.

1. VOLUME RENDERING

In the literature we can meet a wide scale of visualization methods. They can be categorized as surface based and volumetric solutions. In volume rendering objects are typically represented by volume elements called voxels. A voxel can be viewed as a higher dimensional analogy of a pixel. In contrast to surface rendering, volume rendering aims to display the data directly as a transparent cloudy object, as the interior of the objects is always non-empty (always exist).

1.1. Basic theory of raytracing based volume rendering. In general there are three different techniques that create two dimensional image from volumetric data. The first technique considers rays that are cast from each pixel in the image plane into the volume data. Since the rays are cast pixel-wise or in the order of the pixels, these techniques are referred to as *image order techniques* or *raytracing based techniques*. In contrast the *object order techniques* map the intensities of the volume data to the image plane. There are *hybrid techniques* which combine image order and object order methods. Each of these techniques has its own benefits and drawbacks. In recent years image order techniques have become popular, due to the development of GPU-s.

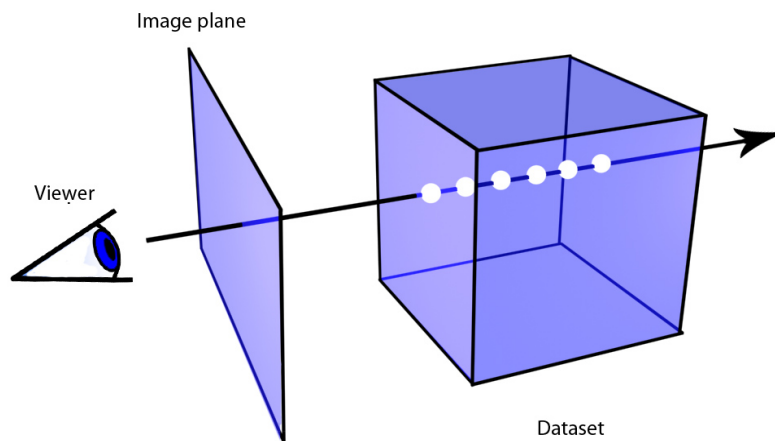


FIGURE 1. Basic idea of image based volume rendering techniques

The algorithm iterates over the pixels of the image plane, where a parameter iterates over the ray. In every iteration step a sample is taken from the volumetric data. The sample value is determined using interpolation inside the volume. The final intensity of a pixel is obtained by integrating through

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the ray using the solution of the following equation, which considers that every voxel emits and absorbs light.

$$\frac{dI}{ds} = q(s)\tau(s) - I(s)\tau(s), \quad (1)$$

where s is the distance along the ray, $I(s)$ is the light intensity, $\tau(s)$ is the absorption coefficient, $q(s)$ is the light emission.

If $s \in [0, \varphi]$, $\varphi \geq 0$ and $I(0) = I_0$ the solution of the equation is [6]:

$$I(s) = I_0 e^{-\int_0^s \tau(\tilde{s}) d\tilde{s}} + \int_0^s q(\tilde{h}) \tau(\tilde{h}) e^{-\int_{\tilde{h}}^s \tau(\tilde{s}) d\tilde{s}} d\tilde{h} \quad (2)$$

The discrete form of the solution is [6]:

$$I = \sum_{i=0}^n q_i \prod_{j=i+1}^n (1 - \alpha_j) \quad (3)$$

Using this solution doesn't result to shadowing in the final image. However, to produce more realistic image the technique needs to be extended.

1.2. Error analysis. For the numerical calculation of the volume rendering integral (formula 3) has rather become preferred. We point out that some other methods with known error bounds can be used to implement the results. We have applied the composite trapezoid rule and Simpson's rule to achieve better results, since no bounds for (3) are available.

The composite trapezoid rule

$$\int_a^b f(x) dx \approx T_n(f) = \frac{h}{2} \left[f(x_1) + 2 \sum_{i=2}^n f(x_i) + f(x_{n+1}) \right] \quad (4)$$

Where $h = \frac{b-a}{n}$. The composite trapezoid rule can be estimated as follows

$$\left| \int_a^b f(x) dx - T_n(f) \right| \leq \frac{M_2(b-a)h^2}{12} = \frac{M_2(b-a)^3}{12n^2}, \quad (5)$$

where M_2 is the bound for the second derivative of function $f(x)$ on the interval $[a, b]$, provided that $f \in C^2[a, b]$.

The composite Simpson rule

$$\int_a^b f(x) dx \approx S_n(f) = \frac{h}{6} \left[f(x_1) + 2 \sum_{i=2}^n f(x_i) + 4 \sum_{i=1}^n f(x_i + \frac{h}{2}) + f(x_{n+1}) \right] \quad (6)$$

where $h = \frac{b-a}{n}$. The composite Simpson rule error is given by

$$\left| \int_a^b f(x) dx - S_n(f) \right| \leq \frac{M_4(b-a)h^4}{2880} = \frac{M_4(b-a)^5}{2880n^4}, \quad (7)$$

where M_4 is the bound for the fourth derivative of function $f(x)$ on the interval $[a, b]$, provided that $f \in C^4[a, b]$.

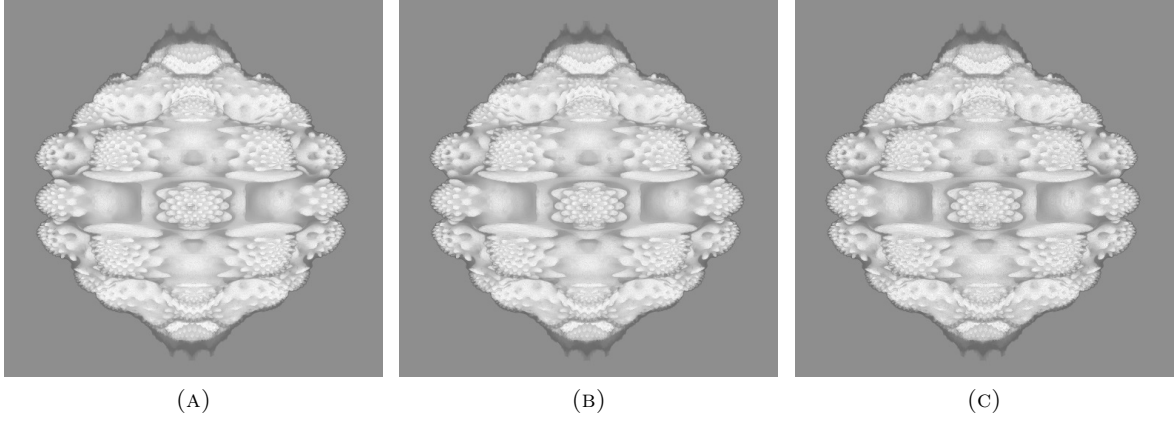


FIGURE 2. Images rendered from the Mandelbulb fractal with different numerical computations of the volume rendering integral (see equation 2). In (a) the method from the literature (see equation 3), in (b) the composite trapezoid rule, (c) the composite Simpson's rule.

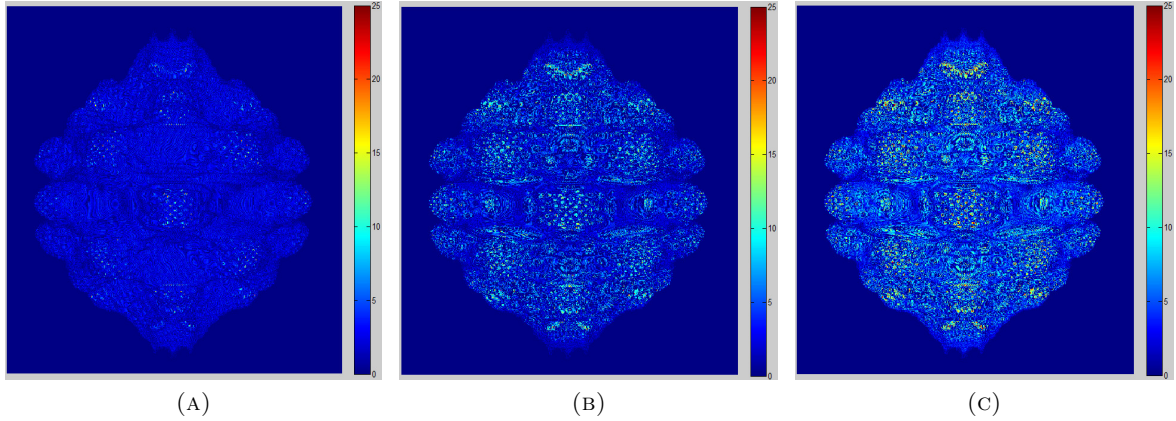


FIGURE 3. In this figure we can see the pixel-wise error calculated from the combination of pictures in Figure 2. The Pictures (a), (b) and (c) of Figure 3 are the respective results of the pixel-wise subtraction of pictures (a) and (b), (b) and (c), and (a) and (c) of Figure 2

1.3. Ambient occlusion. Hernell [7] has proposed a method for computing shadows using a technique called *Ambient Occlusion*. Ambient occlusion is a shading method which takes into account attenuation of light due to occlusion. This technique is based on shooting rays in several directions from each non-transparent voxel in the data set. The rays are terminated at predefined radial boundary making this technique a local approach. The light intensity I_k arriving at the voxel location x from the k th direction is described by the following formula (since the light can arrive from different directions):

$$I_k = \int_a^{R_{\Omega(x)}} \frac{w_k}{R_{\Omega(x)} - a} e^{-\int_a^s \tau(u) du} ds, \quad (8)$$

where a is the translation along the ray, R_{Ω} is the radius of the sphere, w_k weight, $\tau(u)$ is the absorption coefficient. The formula in discrete form:

$$I_k = \sum_{m=0}^M \frac{w_k}{M} \prod_{i=0}^{m-1} (1 - \alpha_i), \quad (9)$$

where M is the number of the samples and α_i is the opacity. The local ambient occlusion for the voxel is given by the sum of all incident light rays I_k .

$$I = \frac{1}{K} \sum I_k, \quad (10)$$

where K is the number of local rays.

In the next section we present the algorithm used to select the "best" sample from a data of samples. This method has been efficiently used in volumetric ambient occlusion, because one can select the "best" sample if more Ambient Occlusion samples are simulated from the local environment. It is worth pointing out that always only one sample can be used at a time in Ambient Occlusion Sampling process.

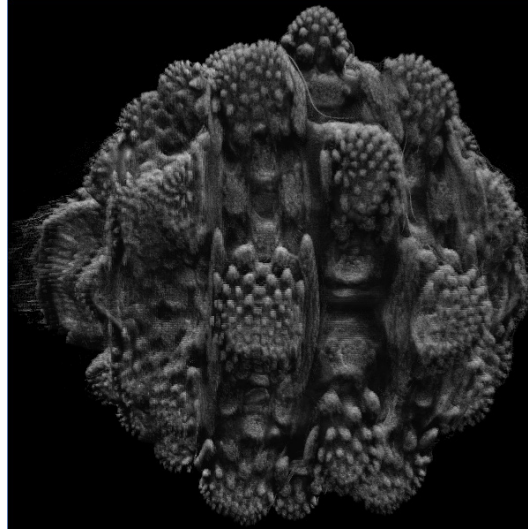


FIGURE 4. Image created from the Mandelbulb fractal using Ambient Occlusion, and optimal averages to select the "best" sample. The size of the data set is 512^3 .

1.4. Algorithmic determination of optimal measure from data. We recall the following useful problem with an algorithm to solve it, proposed in [3]. We should like to note that a similar problem (cf. [8]) has been considered and solved in a different context. Given k samples of n elements with corresponding sample means, our algorithm consists of generating from the data some optimal measure with which we can choose the "best possible" sample. The notion of optimal measure and average will be briefed at the end the communication.

Problem. Let $(\Omega, \mathcal{F})$ be the measurable space with $\Omega = \{1, \dots, n\}$ and $\mathcal{F} = 2^\Omega$, i.e., $\mathcal{F}$ is the power set of Ω . Write $B_1 := \{1\}, \dots, B_n := \{n\}$ and let f be a random variable assuming the theoretical values in $[0, \infty)$. Observe k times this measurable function with results $f_1, \dots, f_k$, i.e.,

B_1	B_2	$\dots$	B_n	
$f_1(1)$	$f_1(2)$	$\dots$	$f_1(n)$	Q_1
$f_2(1)$	$f_2(2)$	$\dots$	$f_2(n)$	Q_2
$\vdots$	$\vdots$		$\vdots$	$\vdots$
$f_k(1)$	$f_k(2)$	$\dots$	$f_k(n)$	Q_k

where

$$Q_i := \frac{1}{n} \sum_{j=1}^n f_{ij}, \quad (i = 1, \dots, k).$$

An algorithm to solve the above problem

The Inputs: n positive integer
 $\Omega = \{1, \dots, n\}$ and X = the power set of Ω
 $k \times n$ matrix $F = [f(i, j)]_{i,j=1}^{n,k}$
 n -dimensional vector Q ; error bound ε
 $B_j = \{j\}, j = 1 \dots n$
Generate the set σ of all permutations of $\{1, \dots, n\}$.

Step 1.

Generate a decreasing sequence
 $\alpha(j) \in (0, 1]$, with $\alpha(1) = 1$.

Step 2.

For any permutation $\{n_1, \dots, n_n\} \in \sigma$

Put $p(B_j) = \alpha(n_j)$, for $j = 1, \dots, n$

Compute the optimal average:

$$A(i) = \max\{f(i, j) * p(B_j) : j = 1, \dots, n\} \text{ for } i = 1, \dots, k$$

Compute the corresponding error: $err = \sqrt{\left(\sum_{j=1}^n (Q(i) - A(i))\right)^2}$

Step 3.

If $err < \varepsilon$ for some permutation then

Find the index i_0 such that $|Q(i_0) - A(i_0)| = \max\{|Q(i) - A(i)| : i = 1, \dots, k\}$

Determine $p(B) = \max\{\alpha(n_j) : j \in B\}$ for each $B \in X$

Else GOTO Step 1

The outputs:

1.) **Best sample:** $f(i_0, 1), \dots, f(i_0, n)$

2.) **The approximated optimal measure:**

2^Ω	$p(B)$
$\{\}$	0
B_1	$p(B_1)$
$\vdots$	$\vdots$
B_i	$p(B_i)$
$\vdots$	$\vdots$

Definition 1.1 (cf. [1]). A set function $p : \mathcal{F} \rightarrow [0, 1]$ will be called optimal measure if it satisfies the following three axioms:

Axiom 1. The identities $p(\Omega) = 1$ and $p(\emptyset) = 0$ hold.

Axiom 2. For all measurable sets B and E , we have $p(B \cup E) = p(B) \vee p(E)$.

Axiom 3. Function p is continuous from above, i.e. whenever $(E_n) \subset \mathcal{F}$ is a decreasing sequence, then

$$p\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n \rightarrow \infty} p(E_n) = \bigwedge_{n=1}^{\infty} p(E_n).$$

Definition 1.2 (cf. [2]). By a p -atom we mean a measurable set H , $p(H) > 0$ such that whenever $B \in \mathcal{F}$ and $B \subset H$, then $p(B) = p(H)$ or $p(B) = 0$.

Definition 1.3 (cf. [2]). A p -atom H is decomposable if there exists a subatom $B \subset H$ such that $p(B) = p(H) = p(H \setminus B)$. If no such subatom exists, we shall say that H is indecomposable.

Theorem 1.1 (The Structure Theorem). Let p be an optimal measure defined on the measurable space $(\Omega, \mathcal{F})$. Then there exists a collection $\mathcal{H}(p) = \{H_n : n \in J\}$ of disjoint indecomposable p -atoms, where J is some countable (i.e. finite or countably infinite) index set, such that for every measurable set $B \in \mathcal{F}$ with $p(B) > 0$ we have

$$p(B) = \bigvee_{n \in J} p(B \cap H_n). \quad (11)$$

Moreover, if the index set J is countably infinite, then the only limit point of the set $\{p(H_n) : n \in J\}$ is 0.

Next, we shall recall the definition of the optimal average or integral as follows.

Let

$$s = \sum_{i=1}^n b_i \chi(B_i)$$

be an arbitrary nonnegative measurable simple function, where $\{B_i : i = 1, \dots, n\} \subset \mathcal{F}$ is a partition of Ω .

Definition 1.4 (cf. [1]). *The quantity*

$$\bigwedge_{\Omega} s dp := \bigvee_{i=1}^n b_i p(B_i)$$

will be called optimal average of s .

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