

Energy spectrum of inhomogeneous rods with elastic and viscous boundary conditions

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Abstract: In this work, we search for the energy spectrum of a continuous longitudinal vibration of an inhomogeneous rod model. This rod model is a continuous limit of a chain oscillator with a spring and a damper between a fixed wall and one end of the chain oscillator. The model describing the system is a wave equation with viscous and elastic boundary conditions. In order to determine the eigenvalues and mode shapes of the rod, separation of variables (space and time) was applied. This method results in a Sturm-Liouville problem for the spatial part, which is solved by using series solution. The distributions of the kinetic and the potential energy are calculated and the energy spectrum of the rod is determined for constant and exponentially varying stiffness.

1. Introduction

Energy transfer occurs in many natural and engineering process along an especially wide range. Some of the most frequent engineering applications are the vibration reduction systems for machines and buildings [2, 3, 4, 5, 6, 7, 9, 12, 16].

A typical example of a process with multi-scale energy transfer is turbulent flow. The energy cascade model of Richardson [13] describes turbulence. The largest whirls break up into smaller eddies, and the energy dissipation of turbulence happens at the smallest scales through viscosity. This process transfers the turbulent kinetic energy of the flow towards the smallest scales. This phenomenon is phenomenologically described by the mechanistic turbulence model via a binary tree (Fig. 1) of Kalmár-Nagy and Bak [10].

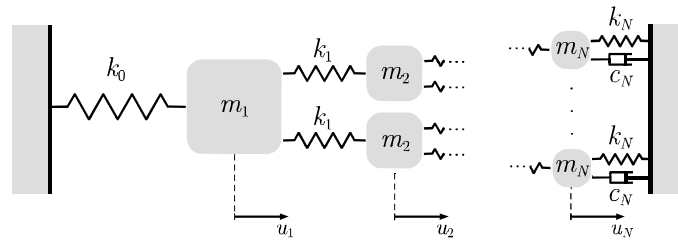


Figure 1. The n -level binary tree model

The energy spectrum $\hat{E}$ describes how the energy is exactly distributed in the different scales of the whirls. It can be defined as a distribution function along the κ wave number:

$$E = \int \hat{E}(\kappa) d\kappa. \quad (1)$$

This definition was applied by Kalmár-Nagy and Bak (2019) [10] and the wavenumber was defined as an inversely proportional quantity to the length-scale of a whirl: $\kappa \sim 1/L$. In case of the binary tree model the wavenumber can be defined as an analogy of the length-scales of the whirls: $\kappa_l = 1/m_l$ (the wavenumber of a certain level is inversely proportional to the mass scale of the level). The different parameters of the mechanistic binary tree model were chosen by Kalmár-Nagy and Bak [10] so as to obtain an energy spectrum similar to the Kolmogorov spectrum [11]:

- A certain level contains only masses of the same size and the sum of the masses on a certain level equals to 1.
- The stiffnesses of the springs are described by a power law as a function of the level.

The binary tree model can be ‘renormalized’ to the N -mass chain oscillator model shown in Fig. 2 [10].

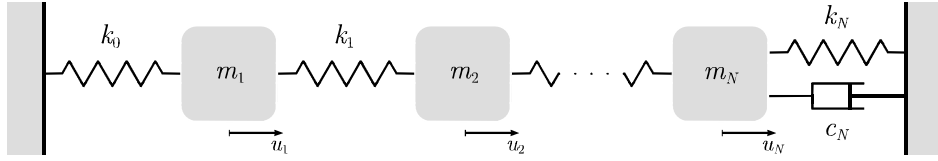


Figure 2. The N -level chain-oscillator model

The parameters of the damped chain oscillator model are the following: m_i is the weight of the i th mass, k_i is the stiffness of the i th spring for $i = 1, 2, \dots, N$ and c_N is the damping. Let the position of the masses in Figure 2 be $u_i = u_i(t)$ for $i = 1, 2, \dots, N$; additionally, the $i = 0$ and $i = N + 1$ cases represent the positions of the fixed ends ($u_0 = 0$, $u_{N+1} = 0$). The parameter distribution that has the least error compared to the Kolmogorov-spectrum has been investigated for a discrete chain oscillator model for 20 levels ($N = 20$) by Kalmár-Nagy and Bak (2019) [10]. The aim of this study is to create a continuous model of the chain oscillator ($N \rightarrow \infty$) and investigate the spectrum in case of different parameter distributions.

2. Derivation of the continuous model

In this section we derive the equations of motion of a continuous (rod) model (Figure 3), which is the limit of the chain oscillator regarding the number of masses ($N \rightarrow \infty$). These

equations contain time and spatial derivatives. The notation for the time derivative is $\dot{\square} = \frac{\partial}{\partial t}\square$ and for the spatial derivative is $\square' = \frac{\partial}{\partial x}\square$. The equations of motion for the system are:

$$m_i \ddot{u}_i = f(\delta_{i+1}, k_{i+1}) - f(\delta_i, k_i), \quad \text{for } i = 1, 2, \dots, N, \quad (2)$$

where f [N] is the force ($f > 0$ when the spring is under tension):

$$f(k, \delta) = k\delta, \quad (3)$$

δ [m] is the contraction of the spring (compared to its tensionless length):

$$\delta_i = u_i - u_{i-1} \quad (4)$$

and k [N/m] is the stiffness of the spring.

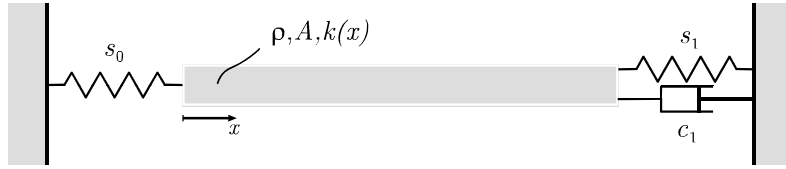


Figure 3. The continuous rod-model of the discrete chain-oscillator

In order to reduce the number of parameters, it is worth introducing dimensionless quantities:

$$\hat{u} = \frac{u}{U}, \quad \hat{f} = \frac{f}{F}, \quad \hat{k} = \frac{k}{K}, \quad \hat{\delta} = \frac{\delta}{\Delta}, \quad \hat{t} = \frac{t}{T}, \quad (5)$$

where u (position), f (force), k (stiffness), δ (displacement), t (time) are dimensional quantities, $\hat{u}, \hat{f}, \hat{k}, \hat{\delta}, \hat{t}$ are their dimensionless representations. These dimensionless quantities are defined by using the following scales: U (position), F (force), K (stiffness), Δ (displacement), T (time) are the *scales* that create connection between the dimensional and non-dimensional quantities. The governing equations of the system are the following:

$$m_i \frac{d^2 u_i}{dt^2} = m_i \frac{d^2 (U \hat{u}_i)}{d(T \hat{t})^2} = K \Delta \left(\hat{f}(\hat{\delta}_{i+1}, \hat{k}_{i+1}) - \hat{f}(\hat{\delta}_i, \hat{k}_i) \right), \quad \text{for } i = 1, 2, \dots, N, \quad (6)$$

which can be written in the following form:

$$\frac{d^2 \hat{u}_i}{d\hat{t}^2} = \frac{T^2 K \Delta}{U m_i} \left(\hat{f}(\hat{\delta}_{i+1}, \hat{k}_{i+1}) - \hat{f}(\hat{\delta}_i, \hat{k}_i) \right). \quad (7)$$

For simplicity a new notation is introduced: the dimensionless parameters (with hat on themselves) will be written without hat, since all of the quantities are dimensionless along the remaining part of the paper. In order to minimize the number of parameters the following combination of the dimensionless quantities are used:

$$\frac{\Delta}{U} = N, \quad \frac{T^2 K}{m_i} = 1. \quad (8)$$

Based on Rosales (2001) [14], the model can be made continuous on the $x \in [0, 1]$ interval. Discrete quantities with index i are changed to continuous quantities $\hat{u}(x), \hat{\delta}(\hat{u}(x))$. The equations of motion for a rod with $k(x)$ stiffness distribution can be written as

$$\ddot{u}(x, t) = \frac{\partial}{\partial x} [k(x)u'(x, t)], \quad (9)$$

with boundary conditions

$$k(0)u'(0, t) = s_0 u(0, t), \quad (10)$$

$$k(1)u'(1, t) = -s_1 u(1, t) - c_1 \dot{u}(1, t), \quad (11)$$

considering the force equilibrium at the ends of the rod. Because of the continuity of the stiffness distribution we have

$$s_0 = \lim_{x \rightarrow 0} k(x), \quad s_1 = \lim_{x \rightarrow 1} k(x). \quad (12)$$

The 2 initial conditions are specified

$$u(x, 0) = \varphi(x), \quad (13)$$

$$u'(x, 0) \equiv 0. \quad (14)$$

The shape $\varphi(x)$ will be determined later in Section 3.

3. Solving the model equations

The boundary value problem (9)–(11) can also be formulated as the Sturm-Liouville problem:

$$\ddot{u}(x, t) = \frac{\partial}{\partial x} [k(x)u'(x, t)], \quad (15)$$

$$k(0)u'(0, t) = s_0 u(0, t), \quad (16)$$

$$k(1)u'(1, t) = -s_1 u(1, t) - c_1 \dot{u}(1, t). \quad (17)$$

To determine the eigenvalues and eigenfunctions of the system the separation of variables method is applied (Hull, 1994) [8]:

$$u(x, t) = U(x)T(t). \quad (18)$$

Equation (18) is substituted into Equation (15):

$$\frac{\ddot{T}(t)}{T(t)} = \left(\frac{U''(x)}{U(x)}k(x) - \frac{U'(x)}{U(x)}k'(x) \right) = \lambda^2, \quad (19)$$

from which

$$T(t) = \tau e^{\lambda t}, \quad (20)$$

$$\frac{\partial}{\partial x} [k(x)U'(x)] + \lambda^2 U(x) = 0. \quad (21)$$

At this point, we consider two special cases, where $k(x)$ is a constant function and where $k(x)$ is an exponentially varying function.

3.1. The $k(x) = \text{const.}$ case

The simplest case of the system is $k(x) = k = \text{const.}$ case. The eigenvalues and the eigenfunctions (mode-shapes) of the rod can be determined in this case in a simple manner. Substituting the constant $k(x)$ into Equation (21) yields

$$kU''(x) + \lambda^2 U(x) = 0, \quad (22)$$

which has the solution

$$U(x) = C_1 e^{\frac{\lambda}{\sqrt{k}}x} + C_2 e^{-\frac{\lambda}{\sqrt{k}}x}, \quad (23)$$

where C_1 and C_2 are constants, which can be determined using the boundary conditions:

$$U'(0) = U(0), \quad (24)$$

$$U'(1) = U(1) - \frac{c_1}{k} \lambda U(1). \quad (25)$$

The following matrix equation results from Equations (23)–(25):

$$\begin{pmatrix} 1 - \frac{\lambda}{\sqrt{k}} & 1 + \frac{\lambda}{\sqrt{k}} \\ e^{\frac{\lambda}{\sqrt{k}}} \left(k + (c_1 + \sqrt{k})\lambda \right) & e^{-\frac{\lambda}{\sqrt{k}}} \left(k + (c_1 - \sqrt{k})\lambda \right) \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (26)$$

Assuming $C_1, C_2 \neq 0$, the determinant of the matrix should be equal to 0:

$$\frac{c_1 \lambda^2 + 2\lambda}{1 + \lambda c_1 + \lambda^2} + \tanh(\lambda) = 0. \quad (27)$$

Equation (27) is the characteristic equation of the system. For a certain k and c_1 values the eigenvalue pairs can be calculated. Notice, that the characteristic equation (27) has infinitely many eigenvalues. Using these eigenvalues, eigenfunction pairs can be determined from Equation (23). In order to calculate the solution function, it is worth applying the series solution method used by Hull (1994) [8]. A suitable initial condition is the sum of the mode-shapes:

$$\varphi(x) = \sum_{n=-\infty}^{\infty} U_n(x), \quad (28)$$

from which the solution function is

$$u(x, t) = \sum_{n=-\infty}^{\infty} U_n(x) T_n(t). \quad (29)$$

Figure 4a illustrates the first 2 dominant eigenvalue pairs of the system for constant $k = 1$ and varying c_1 (root locus plot). Figure 4b illustrates the first 2 mode-shapes of the system.

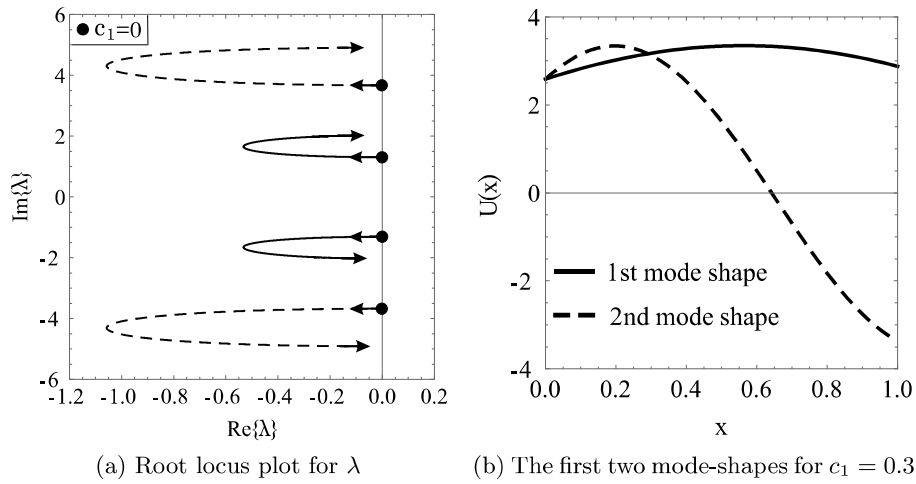


Figure 4. The characteristic roots and the mode-shape for $k = 1$

Following the steps of Hull (1994) [8], the function $u(x, t) = U_{-1}(x)T_{-1}(t) + U_1(x)T_1(t)$ provides a real solution.

3.2. The exponentially changing case

In order to create an analogue model to the binary tree of Kalmár-Nagy and Bak (2019) [10], an exponential term can be used for the stiffness of the rod:

$$k(x) = e^{-2\sigma x}, \quad \sigma \neq 0 \text{ and } x \in [0, 1]. \quad (30)$$

Based on the work of Arfken and Weber (1985) [1, pp. 497-538], the solution of Equation (21) for this case is

$$U(x) = e^{\sigma x} \left[C_1 i I_1 \left(\frac{\lambda e^{\sigma x}}{\sigma} \right) + 2C_2 K_1 \left(\frac{\lambda e^{\sigma x}}{\sigma} \right) \right], \quad (31)$$

where $I_1(z)$ is the first modified Bessel function of the first kind and $K_1(x)$ is the first modified Bessel function of the second kind. The C_1 and C_2 constants can be determined from the boundary conditions (Equations (16)–(17)). The calculation results characteristic equation as in Equation (26)–(27) and real solutions can be determined via series solution method (Equation (29)) as in Section 3.1.

4. Energy spectrum of the rod

In case of the chain oscillator, the energy spectrum is a discrete distribution due to the discrete number of masses, while it is a continuous distribution in the case of the rod model. It is worth collecting the required properties of the ‘wavenumber function’:

- at $x = 0$, the wavenumber is 1
- at $x = 1$ ($N \rightarrow \infty$), the wavenumber is ∞ .

One possible choice is

$$\kappa(x) = 2^{\frac{x}{1-x}}. \quad (32)$$

4.1. Formulation of the energy spectrum

Based on the work of Wohlever and Bernhard (1992) [15], the energy at a point x of the rod can be calculated as

$$E(x, t) = \underbrace{\frac{1}{2} k(x) \left(\frac{du(x, t)}{dx} \right)^2}_{\text{potential energy}} + \underbrace{\frac{1}{2} \left(\frac{du(x, t)}{dt} \right)^2}_{\text{kinetic energy}}. \quad (33)$$

The total energy of the rod is

$$E_{\text{total}}(t) = \underbrace{\frac{1}{2} \int_0^1 k(x) \left(\frac{du(x, t)}{dx} \right)^2 + \left(\frac{du(x, t)}{dt} \right)^2 dx}_{\text{energy of the rod}} + \underbrace{\frac{1}{2} s_0 u^2(0, t) + \frac{1}{2} s_1 u^2(1, t)}_{\text{energy of the springs}}. \quad (34)$$

Substituting back the function of the series solution for the separation of variables method applied from Hull (1994) [8] in Equation (29), Equations (33)–(34) results the energy values. The introduced quantities in Equations (33)–(34) are time-varying quantities, on the

contrary, for calculating the energy spectrum, discrete values are needed at x . In order to define a discrete value at x , it is useful to define a mean energy in the asymptotic limit (Kalmár-Nagy and Bak, 2019) [10]:

$$\bar{E}(x) = \frac{\omega}{\pi} \int_{\tau}^{\tau+\pi/\omega} E(x, t) dt, \quad \tau \rightarrow \infty, \quad (35)$$

$$\bar{E}_{\text{total}} = \frac{\omega}{\pi} \int_{\tau}^{\tau+\pi/\omega} E_{\text{total}}(t) dt, \quad \tau \rightarrow \infty, \quad (36)$$

where ω is the eigenfrequency corresponding to the dominant eigenvalue of the system ($\omega = \text{Im}\{\lambda_1\}$). The same mean energy can be calculated for the total energy, as well; and the fraction of these two mean energies gives the energy spectrum:

$$\hat{E}(x) = \frac{\bar{E}(x)}{\bar{E}_{\text{total}}}, \quad (37)$$

where $\hat{E}(x)$ is the energy spectrum in the range of x (along the rod), $\bar{E}(x)$ is the energy of the rod at x and $\bar{E}_{\text{total}}$ is the total energy of the rod. Figure 5. presents the energy spectrum of the rod model in case of constant stiffness ($k = 1$), and plots can be compared for two different damping values. It can be seen, by increasing the damping, the energy spectrum shifts downwards. There is no significant change in the characteristics of the distribution. The spectrum of the kinetic and potential energy are equal to each other.

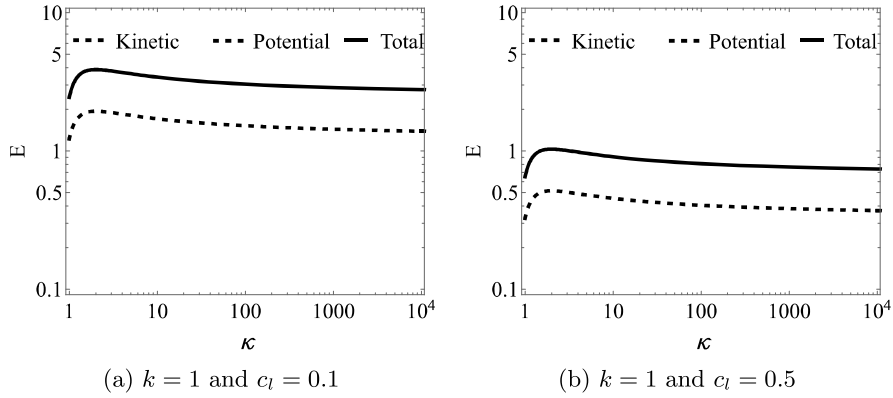


Figure 5. The energy spectrum of the model with constant stiffness for different values

It is worth checking the case where the end-springs have different stiffness, than they were decided on in Equation (12). Figure 6 represents these cases for $k = 1$, $c_1 = 0.5$ and $s_0 s_1 = 1$.

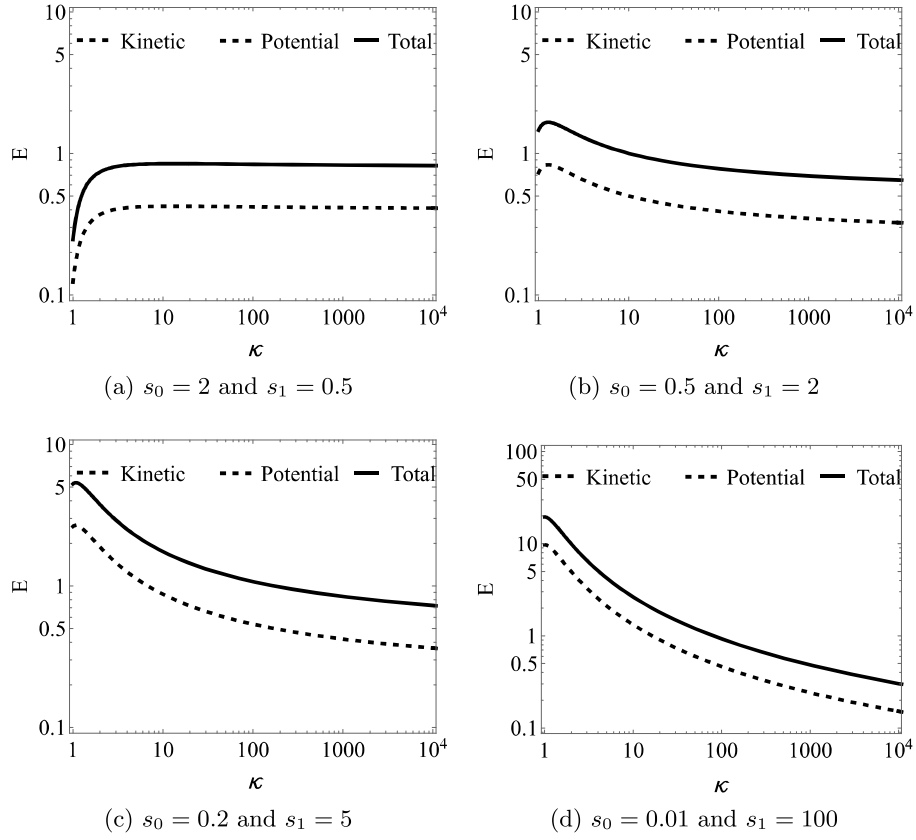


Figure 6. The energy spectrum of the model with constant stiffness for $k = 1$ and $c_1 = 0.5$ and different stiffness values for the end-springs

Figure 6 shows, that increasing s_1 results in a decreasing slope in the energy spectrum. In this case the constant part of the energy spectrum occurs for larger κ values. This constant value is decreasing by increasing s_1 . It can be seen, that decreasing the value of s_0 , the energy value at $\kappa = 1$ is larger.

Figure 7 presents the energy spectra for exponentially varying $k(x)$ in the case of $c_1 = 0.5$ and 4 different σ values. It can be seen, that for $\sigma < 0$, the dominant part of the total energy is the potential energy and for $\sigma > 0$, the dominant part of the total energy is the kinetic energy.

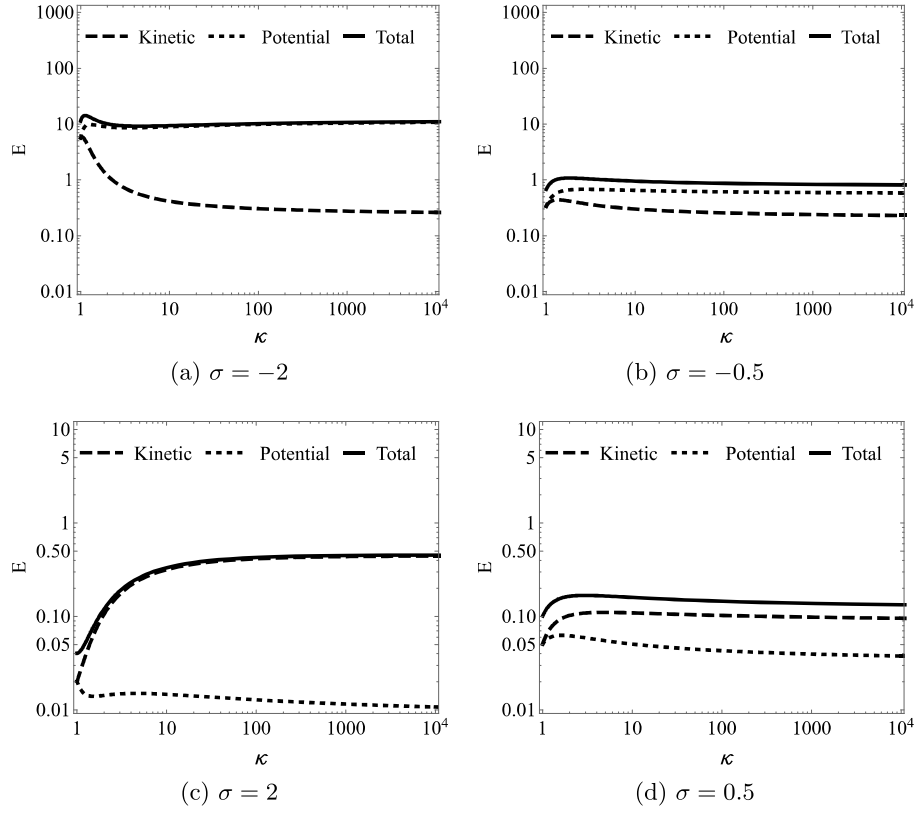


Figure 7. The energy spectrum of the model with exponentially varying stiffness for $c_1 = 0.5$ and different σ values

5. Summary

The chain oscillator model of Kalmár-Nagy and Bak (2019) [10] was extended regarding the number of masses (levels). The new model is continuous, since the number of masses (levels) tends to infinity ($N \rightarrow \infty$). The key parameters of the model is the stiffness distribution and the damping of the system. The eigenvalues and eigenfunctions of the system were determined and the expression for the energy of the rod was derived. The eigenvalues have complex parts, so the system is oscillating. So as to get a discrete value for the energy at a certain wavenumber, it is worth averaging the oscillating energy over its period. Considering a properly long time: after the most of the oscillations are decayed, the energy oscillation can be averaged over an oscillation period connected to the most significant characteristic root. The energy spectrum can be determined for certain $k(x)$ functions, different $c_1 = \text{const.}$ values and different s_0, s_1 stiffness values.

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