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HARMONICALLY EXCITED DELAY EQUATION FOR MACHINE TOOL VIBRATIONS

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ABSTRACT

A machining tool can be subject to different kinds of excitations. The forcing may have external sources (such as rotating imbalance or misalignment of the workpiece) or it can arise from the cutting process itself (e.g. chip formation). We investigate the classical tool vibration model which is a delay-differential equation with a quadratic and cubic nonlinearity and periodic forcing. The method of multiple scales was used to derive the slow-flow equations. The resonance curves of the system are similar to those for the Duffing-equation, having a hardening characteristic. Stability analysis for the fixed points of the slow-flow equations was performed. Local and global bifurcations were studied and illustrated with phase portraits and direct numerical integration of the original equation. Subcritical Hopf, saddle-node and heteroclinic bifurcations were found.

INTRODUCTION

Delay-differential equations are important in many areas of engineering and science. For example in stability analysis, computational techniques (symbolic and numerical), automotive engineering, manufacturing, neuroscience, and control theory (see [1]). Perturbation methods such as the Linstedt-Poincaré method [2], the method of multiple scales [3–5], or the combination of the method of multiple scales and Linstedt-Poincaré method [6] were successfully applied to delay-differential equations.

In this paper the nondimensional form of a harmonically excited one-degree-of-freedom machine tool vibration model is in-

vestigated (for derivation see [7])

$$\ddot{x} + 2\zeta\dot{x} + x = -p(x - x_\tau) + q\left((x - x_\tau)^2 - (x - x_\tau)^3\right) + A \cos(\omega t), \quad (1)$$

where x is the tool displacement ($x_\tau = x(t - \tau)$ is its delayed value with delay $\tau > 0$), $\zeta > 0$ is the relative damping factor, $p > 0$ is the nondimensional cutting force, $q > 0$ is the coefficient of nonlinearity, A is the amplitude of the forcing and ω is its frequency. The regenerative effect is considered with the expressions containing the dimensionless chip thickness variation $(x - x_\tau)$ on the right hand side of Eq. (1).

The stability analysis of the $x = 0$ solution of the linearized equation

$$\ddot{x} + 2\zeta\dot{x} + x = -p(x - x_\tau), \quad (2)$$

was performed in [8, 9]. The characteristic function of (2) is obtained by substituting the trial solution $x(t) = C \exp(i\Omega t)$ into (2). The stability diagram in Figure 1 is given in parametric form (see [10])

$$p = \frac{(1 - \Omega^2)^2 + 4\zeta^2\Omega^2}{2(\Omega^2 - 1)}, \quad (3)$$

$$\tau = \frac{2}{\Omega} \left(j\pi - \arctan \frac{\Omega^2 - 1}{2\zeta\Omega} \right), \quad j = 1, 2, \dots, \quad (4)$$

where j corresponds to the j th ‘lobe’ and $\Omega > 1$. At the minima (‘notches’) of the stability lobes, Ω , p , τ assume the particularly

simple forms

$$\Omega_{crit} = \sqrt{1+2\zeta}, \quad (5)$$

$$p_{crit} = 2\zeta(\zeta+1), \quad (6)$$

$$\tau_{crit} = \frac{2\left(j\pi - \arctan \frac{1}{\sqrt{1+2\zeta}}\right)}{\sqrt{1+2\zeta}}, \quad j = 1, 2, \dots \quad (7)$$

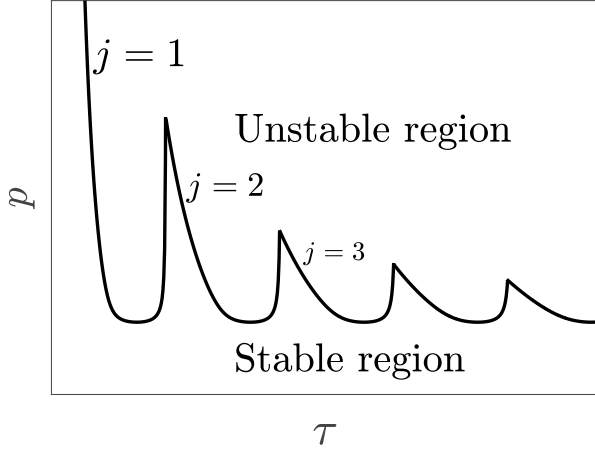


FIGURE 1: The linear stability chart of Eq. (1)

SOLUTION BY THE METHOD OF MULTIPLE SCALES

We will approximate the solution of (1) by using the method of multiple scales. In the following we will assume that damping is small, nonlinearity and forcing are weak (see [10]). In particular

$$\zeta, p, q, A \sim O(\varepsilon), \quad \omega = 1 + \sigma, \quad \sigma \sim O(\varepsilon), \quad (8)$$

where $\varepsilon \ll 1$ is the bookkeeping parameter and σ is the detuning frequency. We express the system parameters as

$$\begin{aligned} \sigma &= \varepsilon \hat{\sigma}, \quad \omega = 1 + \varepsilon \hat{\sigma}, \\ p &= \varepsilon \hat{p}, \quad q = \varepsilon \hat{q}, \quad \zeta = \varepsilon \hat{\zeta}, \quad A = \varepsilon \hat{A}. \end{aligned} \quad (9)$$

Substituting these into (1) we get

$$\ddot{x} + 2\varepsilon \hat{\zeta} \dot{x} + x = -\varepsilon \hat{p}(x - x_\tau) + \varepsilon \hat{q}(x - x_\tau)^2 - \varepsilon \hat{q}(x - x_\tau)^3 + \varepsilon \hat{A} \cos((1 + \varepsilon \hat{\sigma})t). \quad (10)$$

We also assume that the solution of (10) can be well approximated by the two-scale expansion

$$x(t) = x_0(t_0, t_1) + \varepsilon x_1(t_0, t_1) + O(\varepsilon^2), \quad (11)$$

where the timescales are defined as

$$t_0 = t, \quad t_1 = \varepsilon t. \quad (12)$$

With the differential operators

$$D_0 = \frac{\partial}{\partial t_0}, \quad D_1 = \frac{\partial}{\partial t_1}, \quad (13)$$

time differentiation can be written as

$$\frac{d}{dt} = D_0 + \varepsilon D_1 + O(\varepsilon^2), \quad (14)$$

and second derivative with respect to time is

$$\frac{d^2}{dt^2} = D_0^2 + 2\varepsilon D_0 D_1 + O(\varepsilon^2). \quad (15)$$

Substituting the differential operators (14) and (15) into (10) and equating like powers of ε one obtains

$$\varepsilon^0: \quad D_0^2 x_0(t_0, t_1) + x_0(t_0, t_1) = 0, \quad (16)$$

$$\begin{aligned} \varepsilon^1: \quad & D_0^2 x_1(t_0, t_1) + x_1(t_0, t_1) = -2D_0 D_1 x_0(t_0, t_1) \\ & -2\hat{\zeta} D_0 x_0(t_0, t_1) - \hat{p}[x_0(t_0, t_1) - x_0(t_0 - \tau, t_1)] \\ & + \hat{q}[x_0(t_0, t_1) - x_0(t_0 - \tau, t_1)]^2 \\ & - \hat{q}[x_0(t_0, t_1) - x_0(t_0 - \tau, t_1)]^3 + \hat{A} \cos(t_0 + \hat{\sigma} t_1). \end{aligned} \quad (17)$$

Solving (16) for $x_0(t_0, t_1)$ yields

$$x_0(t_0, t_1) = a(t_1) e^{i t_0} + \bar{a}(t_1) e^{-i t_0}. \quad (18)$$

We substitute this solution into (17) and eliminate the secular terms which would give rise to unbounded terms

$$\begin{aligned} 0 &= -\frac{1}{2} \hat{A} e^{i \hat{\sigma} t_1} + 2i D_1 a(t_1) \\ &+ a(t_1) \left[\hat{p}(1 - e^{-i \tau}) + 2i \hat{\zeta} - 3e^{-2i \tau} (e^{i \tau} - 1)^3 \hat{q} a(t_1) \bar{a}(t_1) \right]. \end{aligned} \quad (19)$$

Writing $a(t_1)$ in a polar form (where $\alpha(t_1)$ is the amplitude and $\beta(t_1)$ is the phase)

$$a(t_1) = \frac{1}{2} \alpha(t_1) e^{i \beta(t_1)}, \quad \bar{a}(t_1) = \frac{1}{2} \alpha(t_1) e^{-i \beta(t_1)}. \quad (20)$$

then substituting into (19), we get

$$\begin{aligned} 0 &= -\frac{1}{2} \hat{A} e^{i \hat{\sigma} t_1} + i D_1 \alpha(t_1) e^{i \beta(t_1)} + \frac{1}{2} \alpha(t_1) e^{i \beta(t_1)} \\ &\left[\hat{p}(1 - e^{-i \tau}) + 2i \hat{\zeta} - \frac{3}{4} e^{-2i \tau} (e^{i \tau} - 1)^3 \hat{q} \alpha^2(t_1) \right]. \end{aligned} \quad (21)$$

Now we divide (21) by $e^{i \beta(t_1)}$ and introduce

$$\phi(t_1) = \hat{\sigma} t_1 - \beta(t_1), \quad (22)$$

to get

$$\begin{aligned} 0 &= -\frac{1}{2} \hat{A} e^{i \phi(t_1)} + i D_1 \alpha(t_1) + \frac{1}{2} \alpha(t_1) \\ &\left[\hat{p}(1 - e^{-i \tau}) + 2i \hat{\zeta} - \frac{3}{4} e^{-2i \tau} (e^{i \tau} - 1)^3 \hat{q} \alpha^2(t_1) \right]. \end{aligned} \quad (23)$$

Separating the real and imaginary parts of equation (23) results in two ordinary differential equations describing the evolution of the amplitude and phase

$$D_1 \alpha = - \left(\hat{\zeta} + \frac{\hat{p}}{2} \sin \tau \right) \alpha - 3 \hat{q} \cos \frac{\tau}{2} \sin^3 \frac{\tau}{2} \alpha^3 + \frac{\hat{A}}{2} \sin \phi, \quad (24)$$

$$\alpha D_1 \phi = \frac{1}{2} (\hat{p}(\cos \tau - 1) + 2\hat{\sigma}) \alpha - 3\hat{q} \sin^4 \frac{\tau}{2} \alpha^3 + \frac{\hat{A}}{2} \cos \phi. \quad (25)$$

To get the amplitude of the steady-state primary resonance, i.e. the fixed points of (24) and (25) we set the left-hand sides of (24) and (25) to zero and solve the resulting algebraic equations (by eliminating the trigonometric terms from them). From these for the amplitude α^* and for the phase ϕ^* we get

$$\left[\frac{1}{4} (\hat{p}(1 - \cos \tau) - 2\hat{\sigma} + 6\hat{q}\alpha^{*2} \sin^4 \frac{\tau}{2})^2 + \left(\hat{\zeta} + \frac{\hat{p}}{2} \sin \tau + 3\hat{q}\alpha^{*2} \cos \frac{\tau}{2} \sin^3 \frac{\tau}{2} \right)^2 \right] \alpha^{*2} - \frac{\hat{A}^2}{4} = 0. \quad (26)$$

$$\phi^* = \arcsin \left[\frac{2}{\hat{A}} \left(\left(\hat{\zeta} + \frac{\hat{p}}{2} \sin \tau \right) \alpha^* + 3\hat{q} \cos \frac{\tau}{2} \sin^3 \frac{\tau}{2} \alpha^{*3} \right) \right]. \quad (27)$$

Now we rewrite these equations using the original ζ, p, q, A, ω parameters. We substitute $\hat{\sigma} = (\omega - 1)/\varepsilon$ into (26) and multiply the expression by ε^2 to get

$$\left[\frac{1}{4} (p(1 - \cos \tau) - 2(\omega - 1) + 6q\alpha^{*2} \sin^4 \frac{\tau}{2})^2 + \left(\zeta + \frac{p}{2} \sin \tau + 3q\alpha^{*2} \cos \frac{\tau}{2} \sin^3 \frac{\tau}{2} \right)^2 \right] \alpha^{*2} - \frac{A^2}{4} = 0. \quad (28)$$

And the expression for the phase with the original parameters is

$$\phi^* = \arcsin \left[\frac{2}{A} \left(\left(\zeta + \frac{p}{2} \sin \tau \right) \alpha^* + 3q \cos \frac{\tau}{2} \sin^3 \frac{\tau}{2} \alpha^{*3} \right) \right]. \quad (29)$$

Equation (28) is a cubic equation for α^{*2} , which can be solved for α^* resulting 1, 2 or 3 pair ($\pm$) of real roots.

STEADY STATE AMPLITUDES FOR DIFFERENT FORCING AMPLITUDES

The shape of α^* as a function of ω is illustrated in Figure 2 for $A = \{0.001, 0.003, 0.005, 0.007, 0.009\}$.

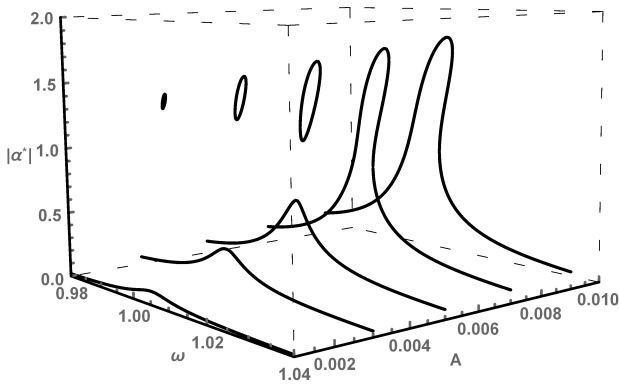


FIGURE 2: The fixed point $\alpha^*(\omega)$ for different A values.

For some parameter values the amplitude response consists of two disjoint curves, similarly as in [11].

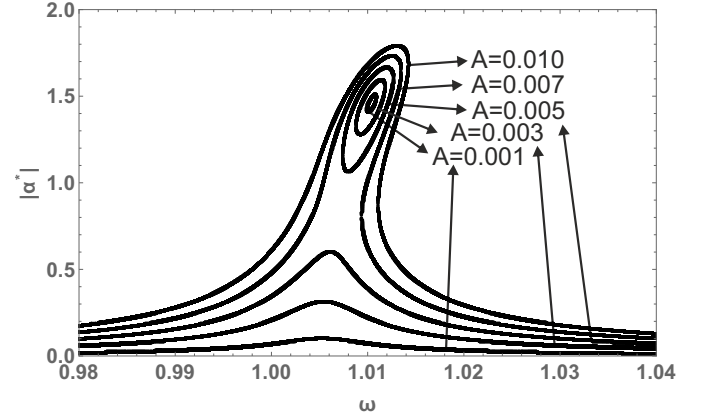


FIGURE 3: The fixed point α^* as a function of ω for different A values.

ANALYSIS OF THE SLOW-FLOW

Here we study the slow-flow equations (24) and (25) (the hats disappear when both sides are multiplied by ε). For further investigation we set the parameter values at the first lobe ($j=1$, see Figure 1)

$$\tau = 4.676, \zeta = 0.01, p = 0.5p_{crit} = 0.0101, q = 0.003, A = 0.01. \quad (30)$$

For these values the amplitude α^* as a function of ω is illustrated in Figure 4, which is the uppermost curve of Figure 3.

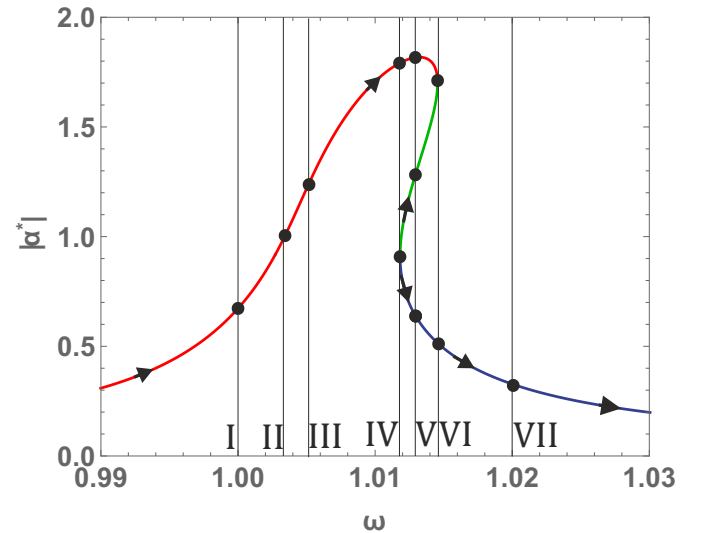


FIGURE 4: The amplitude α^* as a function of ω .

The vertical lines correspond to particular values of ω_i , where $i \in \{I, II, \dots, VII\}$; $\omega_i \in \{1, 1.00364, 1.005, 1.0118, 1.013, 1.0146, 1.02\}$.

We now examine the stability of the fixed points (α^*, ϕ^*) for different forcing frequencies ω . Stability is determined by the characteristic roots λ of the Jacobian

$$J = \begin{pmatrix} -\zeta - \frac{p}{2} \sin \tau - 9q\alpha^{*2} \cos \frac{\tau}{2} \sin^3 \frac{\tau}{2} & \frac{A}{2} \cos \phi^* \\ -6\alpha^* q \sin^4 \frac{\tau}{2} - \frac{A}{2\alpha^{*2}} \cos \phi^* & -\frac{A}{2\alpha^*} \sin \phi^* \end{pmatrix}. \quad (31)$$

Figures 5-7 show root locus plots. On these the bifurcation parameter ω was increased from $\omega = 0.99$ to $\omega = 1.02$. The points with Roman numbers indicate the corresponding $\omega_i \in \{1, 1.00364, 1.005, 1.0118, 1.013, 1.0146, 1.02\}$ values.

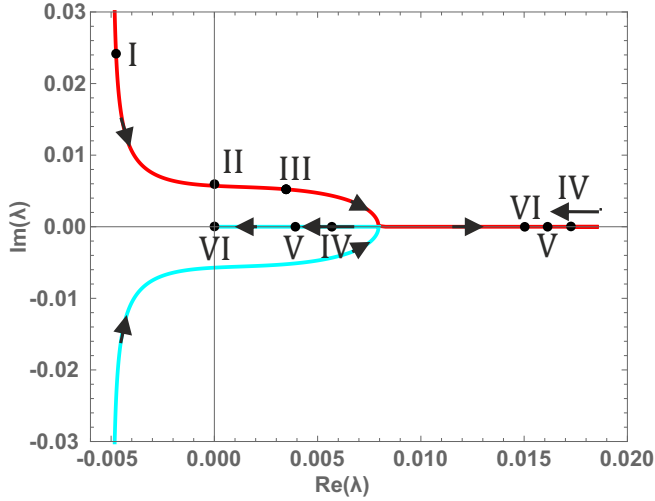


FIGURE 5: Characteristic roots of J for $\omega \in [0.99, \omega_{VI}]$

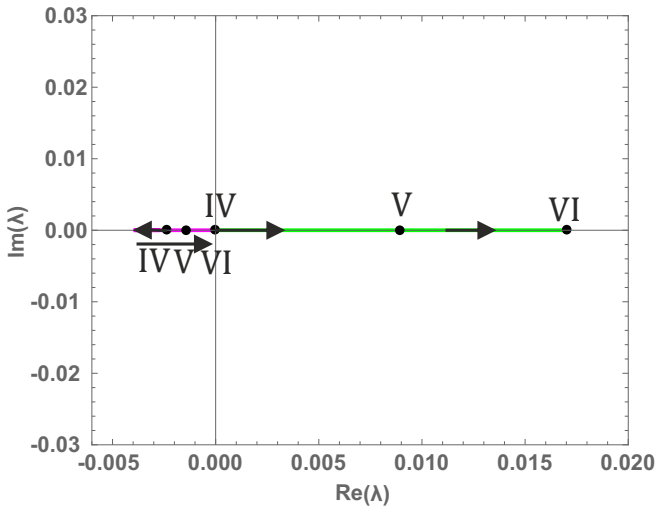


FIGURE 6: Characteristic roots of J for $\omega \in [\omega_{IV}, \omega_{VI}]$

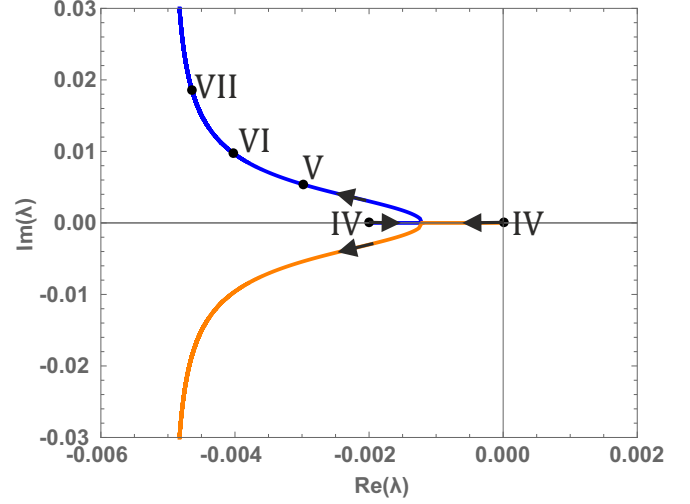


FIGURE 7: Characteristic roots of J for $\omega \in [\omega_{IV}, 1.02]$

Figure 8 shows the phase portrait of the slow-flow at ω_I . The filled circle is the stable spiral and the dashed line is the unstable limit cycle. At ω_{II} a pair of complex conjugate characteristic roots cross the imaginary axis. At this point the fixed point is weakly repelling, giving rise to a subcritical Hopf bifurcation (Figure 9). After the Hopf bifurcation point the stable spiral becomes unstable. This is shown in Figure 10, where the empty circle is the unstable spiral at ω_{III} . Further increasing the bifurcation parameter ω the unstable spiral becomes an unstable node. In Figure 11 at ω_{IV} the empty circle at $\alpha^* = 1.72$ is the unstable node, the empty circle at $\alpha^* = 0.93$ is nonhyperbolic fixed point undergoing saddle-node bifurcation. After the saddle-node bifurcation a stable node and a saddle point is created (Figure 12 at ω_V). The stable node transforms into a stable spiral at $\omega = 1.0119$ and the saddle point moves towards the unstable node. At ω_{VI} the unstable node and saddle point coalesce and a heteroclinic bifurcation occurs (Figure 13). After the heteroclinic bifurcation an unstable limit cycle (dashed line) is created (see Figure 14).

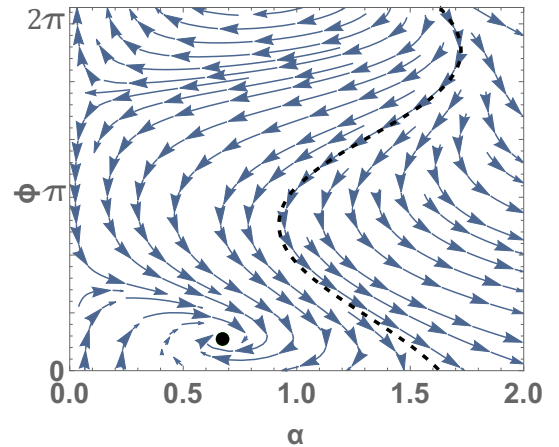


FIGURE 8: Phase portrait at ω_I

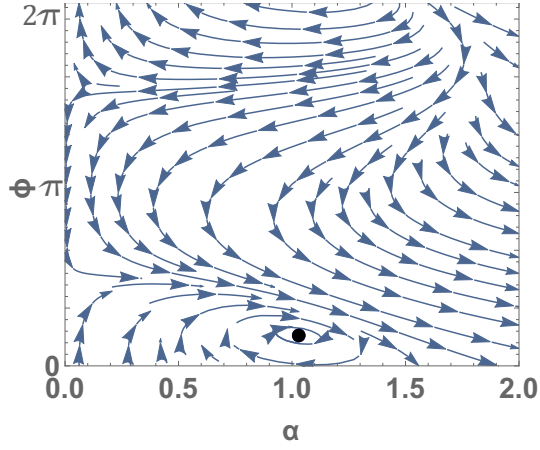


FIGURE 9: Phase portrait at ω_{II}

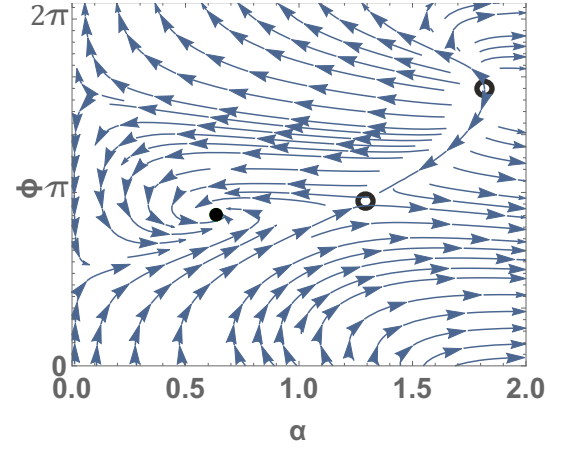


FIGURE 12: Phase portrait at ω_V

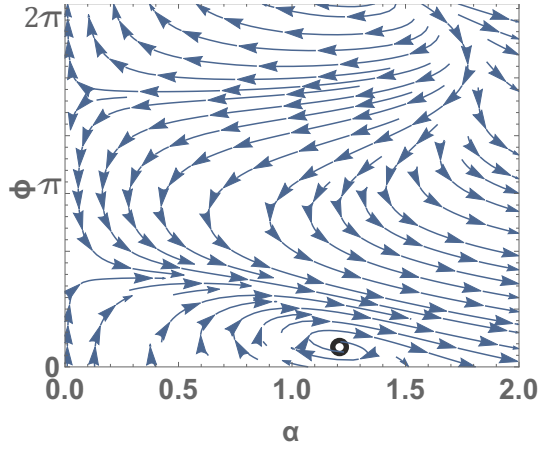


FIGURE 10: Phase portrait at ω_{III}

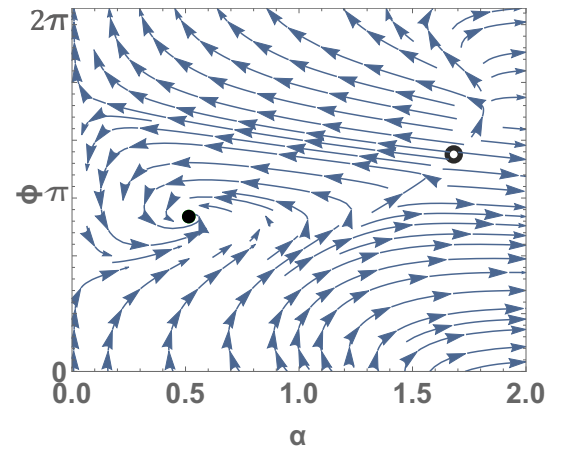


FIGURE 13: Phase portrait at ω_{VI}

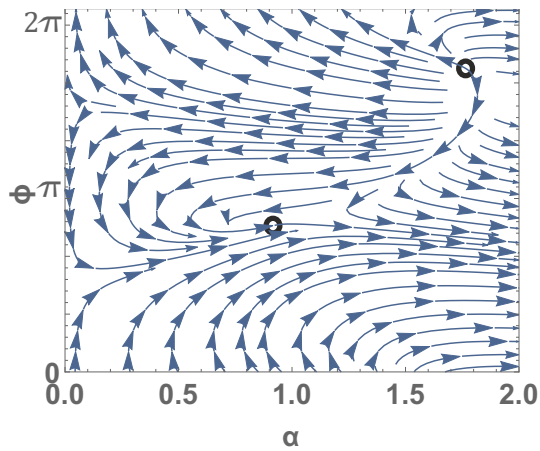


FIGURE 11: Phase portrait at ω_{IV}

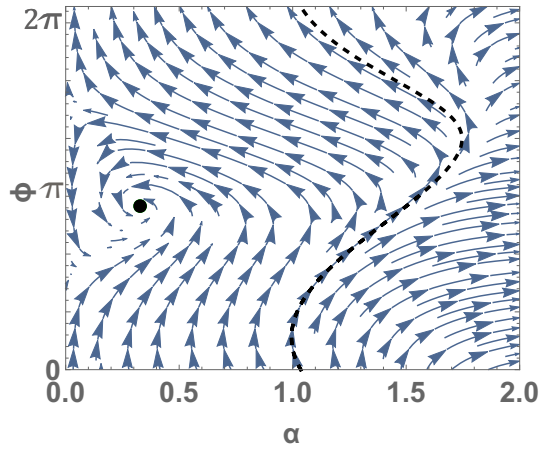


FIGURE 14: Phase portrait at ω_{VII}

NUMERICAL SOLUTION OF THE DELAY EQUATION

The direct numerical solution of (1) with parameters specified in (30) was determined in Mathematica using the Dormand-Prince method with an initial function $\alpha^* \cos(\omega t + \phi^*)$, $t \in$

$[-\tau, 0]$, where α^* and ϕ^* were determined from (28) and (29). The amplitude of the numerical solution was determined in the time interval $[250\tau, 450\tau]$ as

$$\alpha_{num}^* = \max |x(t)|, \quad t \in [250\tau, 450\tau]. \quad (32)$$

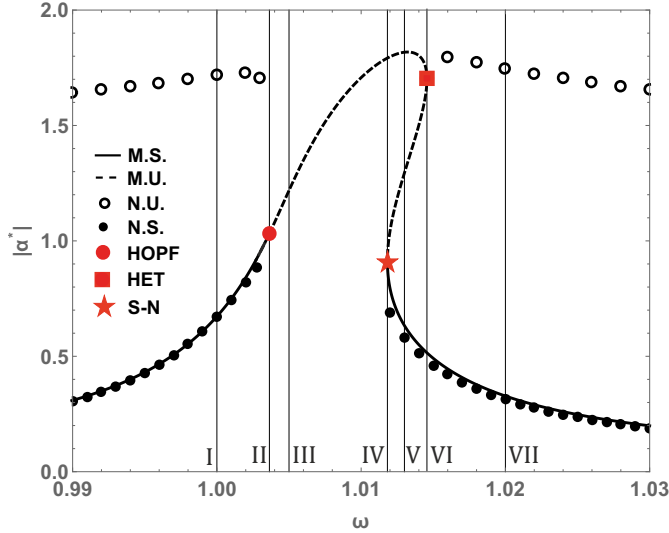


FIGURE 15: Primary resonance curve.

Figure 15 shows the amplitude α^* as a function of forcing frequency ω where the solid line indicates the stable method of multiple scales (M. S.) solution, the dashed line the unstable method of multiple scales (M. U.) solution. The stable numerical (N. S.) equilibrium solutions are shown with filled circles, unstable limit cycles (N. U.) are depicted by empty circles. The thin vertical lines indicate the $\omega_i, i \in \{I, II, \dots, VII\}$ angular frequencies. The red markers correspond to the subcritical Hopf (HOPF), heteroclinic (HET) and saddle-node (S-N) bifurcation points.

Time series and phase plane trajectories for periodic and quasi-periodic solutions for $x(t)$ with excitation frequency $\omega_I = 1$ are shown in Figure 16 and 17.

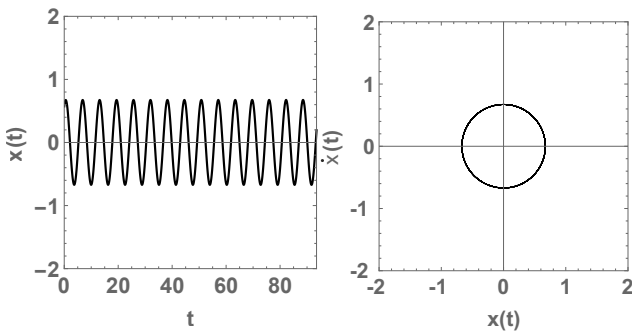


FIGURE 16: Periodic numerical solution started with the initial function $x(t) = 0.6727 \cos(t + 0.5522), t \in [-4.676, 0]$.

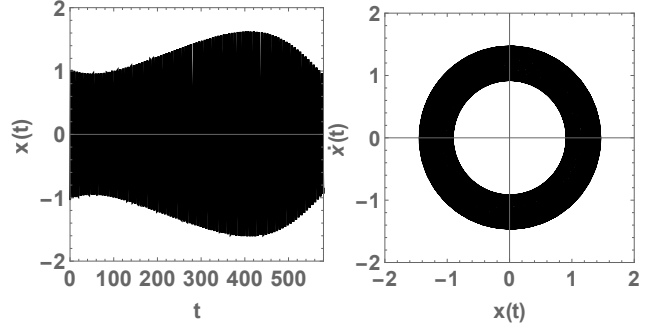


FIGURE 17: Quasi-periodic numerical solution started with the initial function $x(t) = 1.077 \cos(t), t \in [-4.676, 0]$.

CONCLUSIONS

Equation (1) without forcing admits a subcritical Hopf bifurcation [10] and the interaction of forcing and Hopf bifurcation may lead to complex phenomena [12]. Plaut and Hsieh [13] studied a one-DOF mechanical system with delay and excitation and found periodic, chaotic and unbounded responses. Forcing of a Duffing-type equation without delay can also result in complicated bifurcation structure [14, 15].

The method of multiple scales gave an excellent approximation of the solution of the nonlinear delay-differential equation (1). The resonance curves found here are similar to those for the Duffing-equation [16], having a hardening characteristic. We found subcritical Hopf bifurcation, saddle-node bifurcation and heteroclinic bifurcation.

REFERENCES

- [1] Balachandran, B., Kalmár-Nagy, T., and Gilsinn, D. E., 2009. *Delay differential equations*. Springer.
- [2] Casal, A., and Freedman, M., 1980. "A Poincaré-Lindstedt approach to bifurcation problems for differential-delay equations". *IEEE Transactions on Automatic Control*, **25**(5), pp. 967–973.
- [3] Nayfeh, A., Chin, C., and Pratt, J., 1997. "Applications of perturbation methods to tool chatter dynamics". *Dynamics and chaos in manufacturing processes*, pp. 193–213.
- [4] Nayfeh, A. H., 2008. "Order reduction of retarded nonlinear systems—the method of multiple scales versus center-manifold reduction". *Nonlinear Dynamics*, **51**(4), pp. 483–500.
- [5] Oztepe, G. S., Choudhury, S. R., and Bhatt, A., 2015. "Multiple scales and energy analysis of coupled Rayleigh-van der Pol oscillators with time-delayed displacement and velocity feedback: Hopf bifurcations and amplitude death". *Far East Journal of Dynamical Systems*, **26**(1), p. 31.
- [6] Pakdemirli, M., and Karahan, M. M. F., 2010. "A new perturbation solution for systems with strong quadratic and cu-

- bic nonlinearities”. *Mathematical Methods in the Applied Sciences*, **33**(6), pp. 704–712.
- [7] Kalmár-Nagy, T., Stépán, G., and Moon, F. C., 2001. “Subcritical Hopf bifurcation in the delay equation model for machine tool vibrations”. *Nonlinear Dynamics*, **26**(2), pp. 121–142.
 - [8] Hanna, N., and Tobias, S., 1974. “A theory of nonlinear regenerative chatter”. *ASME Journal of Engineering for Industry*, **96**(1974), pp. 247–255.
 - [9] Stépán, G., 1989. *Retarded dynamical systems: stability and characteristic functions*. Longman Scientific & Technical.
 - [10] Kalmár-Nagy, T., 2002. “Delay-differential models of cutting tool dynamics with nonlinear and mode-coupling effects.”. *PhD thesis*.
 - [11] Zhang, Y., and Golubitsky, M., 2011. “Periodically forced Hopf bifurcation”. *SIAM Journal on Applied Dynamical Systems*, **10**(4), pp. 1272–1306.
 - [12] Gambaudo, J.-M., 1985. “Perturbation of a Hopf bifurcation by an external time-periodic forcing”. *Journal of differential equations*, **57**(2), pp. 172–199.
 - [13] Plaut, R., and Hsieh, J.-C., 1987. “Chaos in a mechanism with time delays under parametric and external excitation”. *Journal of Sound and Vibration*, **114**(1), pp. 73–90.
 - [14] Sanchez, N. E., and Nayfeh, A. H., 1990. “Prediction of bifurcations in a parametrically excited Duffing oscillator”. *International Journal of Non-Linear Mechanics*, **25**(2-3), pp. 163–176.
 - [15] Zavodney, L. D., Nayfeh, A., and Sanchez, N., 1990. “Bifurcations and chaos in parametrically excited single-degree-of-freedom systems”. *Nonlinear Dynamics*, **1**(1), pp. 1–21.
 - [16] Nayfeh, A. H., and Mook, D. T., 1979. “Forced oscillations of systems having a single degree of freedom”. *Nonlinear Oscillations*, pp. 161–257.