

Bifurcation Analysis of a Forced Delay Equation for Machine Tool Vibrations

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Abstract. A machining tool can be subject to different kinds of excitation. Harmonic forcing can arise from the cutting process itself (e.g., periodic chip formation). We investigate the classical tool vibration model which is a delay-differential equation with a quadratic and cubic nonlinearity and periodic forcing. The method of multiple scales (MMS) was used to derive the slow flow equations. Two-parameter bifurcation analysis of equilibria and limit cycles of the slow flow equations was performed, using Matcont. Sub- and supercritical Hopf, cusp, fold, Generalized Hopf (Bautin), Bogdanov-Takens bifurcations were found. This bifurcation analysis shows the rich dynamics of the system.

Introduction

Delay-differential equations are essential in many areas of engineering and science. For example in stability analysis, computational techniques (symbolic and numerical), automotive engineering, manufacturing, neuroscience, population dynamics, economics, and control theory (see [1, 2]). Various techniques like the Linstedt-Poincaré method [3], the method of multiple scales [4], the combination of the method of multiple scales and the Linstedt-Poincaré method [5], harmonic balance [6], method of averaging [7], center manifold reduction [8] and normal forms [9] have been used to study delay-differential equations. In this paper, a harmonically excited one-degree-of-freedom machine tool vibration model is investigated (for derivation see [6, 8])

$$\ddot{x} + 2\zeta\dot{x} + x = -p(x - x_\tau) + q\left((x - x_\tau)^2 - (x - x_\tau)^3\right) + A \cos(\omega t), \quad (1)$$

where x is the tool displacement, $\zeta > 0$ is the relative damping factor, $p > 0$ is the nondimensional cutting force, $q > 0$ is the coefficient of nonlinearity, A is the amplitude of the forcing and ω is its frequency. The term x_τ denotes $x(t - \tau)$, the delayed value of the position with positive delay τ .

Slow flow equations of the amplitude and phase

We approximate the solution of Eq. (1) by using the method of multiple scales. We assume that damping is small, nonlinearity and forcing are weak (see [6]), and the forcing is near-resonant, i.e., $\zeta, p, q, A, \sigma \sim O(\varepsilon)$ and $\omega = 1 + \sigma$; where $\varepsilon \ll 1$ is the bookkeeping parameter and σ is the detuning frequency. We express the system parameters using the bookkeeping parameter as $\sigma = \varepsilon\hat{\sigma}$, $\omega = 1 + \varepsilon\hat{\sigma}$, $p = \varepsilon\hat{p}$, $q = \varepsilon\hat{q}$, $\zeta = \varepsilon\hat{\zeta}$, $A = \varepsilon\hat{A}$. We also assume that the solution of Eq. (1) can be well approximated by the two-scale expansion, using the fast $t_0 = t$ and the slow timescale $t_1 = \varepsilon t$ i.e., $x(t) = x_0(t_0, t_1) + \varepsilon x_1(t_0, t_1) + O(\varepsilon^2)$. Using the multiple scales method, the slow flow equations describing the slowly varying amplitude and phase (for derivation see [10]) are

$$\begin{aligned} D_1\alpha &= -\left(\hat{\zeta} + \frac{\hat{p}}{2}\sin\tau\right)\alpha - 3\hat{q}\cos\frac{\tau}{2}\sin^3\frac{\tau}{2}\alpha^3 + \frac{\hat{A}}{2}\sin\phi, \\ \alpha D_1\phi &= \frac{1}{2}(\hat{p}(\cos\tau - 1) + 2\hat{\sigma})\alpha - 3\hat{q}\sin^4\frac{\tau}{2}\alpha^3 + \frac{\hat{A}}{2}\cos\phi, \end{aligned} \quad (2)$$

where $D_1 = \partial/\partial t_1$. The amplitude $\alpha(t_1) > 0$ and the phase $\phi(t_1)$ come from the solution of $x_0(t_0, t_1)$, i.e., $x_0(t_0, t_1) = \alpha(t_1)\cos(\omega t_0 - \phi(t_1))$.

Bifurcation analysis of the slow flow

In this section, we will summarize the bifurcation analysis of equilibria and limit cycles with Matcont [11]. The forcing amplitude A and frequency ω are the two bifurcation parameters. The other system parameters were the following during the investigation $\tau = 4.676$, $\zeta = 0.01$, $p = 0.0101$, $q = 0.003$ (see [8, 10]). Using Matcont we determined two essential bifurcation curves.

The first one is the fold bifurcation curve, with three cusp bifurcation points and two Bogdanov-Takens points. Inside the closed wedge region (filled with grey in Figure 1a) of the fold curve three equilibrium points exist, for the other areas of the (A, ω) plane we have one equilibrium point (see Figure 1a). The number of equilibrium points corresponds to different amplitudes of the steady-state vibration.

The second one is the Hopf-curve, with a Generalized Hopf (Bautin) bifurcation point and two Bogdanov-Takens points (see Figure 1b). At the Generalized Hopf bifurcation point the first Lyapunov coefficient vanishes. This bifurcation point separates branches of subcritical and supercritical Hopf bifurcations (blue and green lines in Figure 1b, 1c). For nearby parameter values, the fold bifurcation of limit cycles occur [12]. The limit cycles of the sub- and supercritical Hopf bifurcations are the quasi-periodic solutions of Eq. (1).

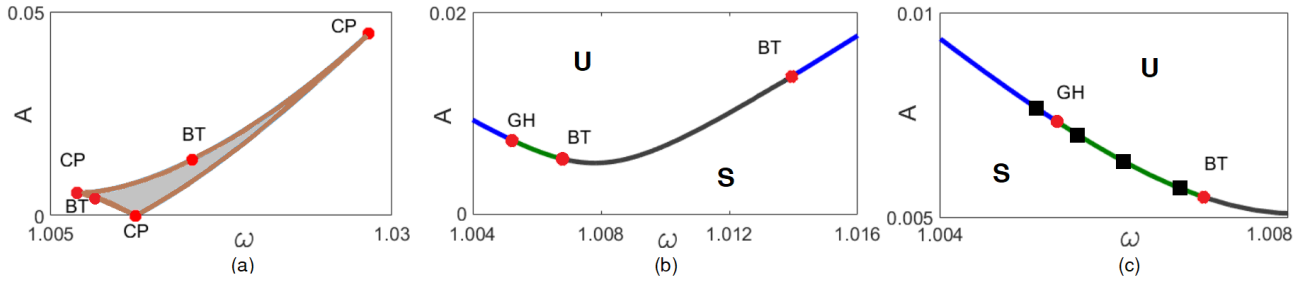


Figure 1: Bifurcation diagrams (CP - cusp, BT - Bogdanov-Takens, GH - Generalized Hopf, S - stable region, U - unstable region) We also investigate the vicinity of the Generalized Hopf bifurcation. To illustrate the bifurcations we chose 4 points on the Hopf Curve, one before and three after the Generalized Hopf bifurcation point. Squares denote these points in Figure 1c.

We made a limit cycle continuation using the parameter ω and the Period in Matcont. At the first point $A = 0.0075$ the Hopf bifurcation is subcritical. After the Generalized Hopf bifurcation point the Hopf bifurcation becomes supercritical, and the system has two limit cycles which collide and disappear in a fold bifurcation, e.g., $A = 0.0070$ and $A = 0.0065$. Approaching the Bogdanov-Takens point the system has one limit cycle again, e.g., $A = 0.0060$. These limit cycles are illustrated in 2D and also in 3D in Figure 2.

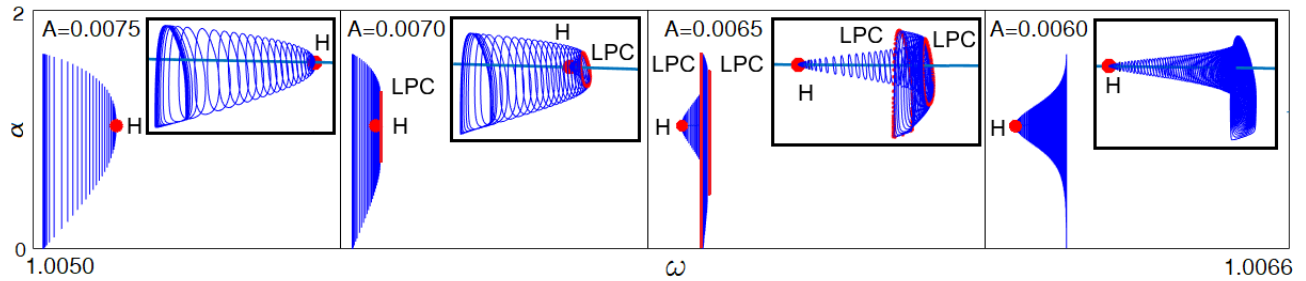


Figure 2: Limit cycles before and after the Generalized Hopf bifurcation

Conclusions

Equation (1) without forcing admits a subcritical Hopf bifurcation [8], and the interaction of forcing and Hopf bifurcation leads to complex phenomena. The two-parameter bifurcation analysis of the slow flow equations has been shown. Sub- and supercritical Hopf, cusp, fold, Generalized Hopf (Bautin), Bogdanov-Takens bifurcations were found. This demonstrates the rich dynamics of the system.

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