

Bifurcation analysis of a forced delay equation for machine tool vibrations

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Abstract A machining tool can be subject to different kinds of excitations. The forcing may have external sources (such as rotating imbalance, misalignment of the workpiece or ultrasonic excitation), or it can arise from the cutting process itself (e.g. periodic chip formation).

We investigate the classical one degree of freedom tool vibration model, a delay-differential equation with quadratic and cubic nonlinearity, and periodic forcing. The method of multiple scales is used to derive the slow flow equations. Stability and bifurcation analysis of equilibria of the slow flow equations is presented. Analytical expressions are obtained for the saddle-node and Hopf bifurcation points. Bifurcation analysis is also carried out numerically. Sub- and supercritical Hopf, cusp, fold, generalized Hopf (Bautin), Bogdanov-Takens bifurcations are found. Limit cycle continuation is performed using MatCont.

Local and global bifurcations are studied and illustrated with phase portraits and direct numerical integration of the original equation.

Keywords Time delay · Retarded systems · Multiple scales · Bifurcation · MatCont

1 Introduction

Delay-differential equations appear in many areas of engineering and science. For example in automotive engineering, manufacturing, neuroscience, population dynamics, economics, and control theory (see [1–4]).

Various techniques like the method of multiple scales [5–10], the Linstedt-Poincaré method [11], the combination of the method of multiple scales and the Linstedt-Poincaré method [12], harmonic balance method [13–15], method of averaging [16], semi-discretization [17], center manifold reduction [18, 19] and normal forms [20] have been used to study delay-differential equations.

Our focus here is harmonically forced delay-differential equations. These type of equations have been studied in the past decades [21–31]. In particular, we will study machine tool vibrations with external forcing. Tool vibrations are described by delay-differential equations, since the chip thickness, which determines the cutting force, is affected both by actual and delayed tool positions due to the surface regeneration effect [32–37].

To better understand the machining process, the chip formation has to be modeled. Periodic chip formation was investigated in [38–46]. In recent works of Csernák and Pálmai [47, 48], a nonlinear system of differential equations is used to model chip segmentation. They find periodic, aperiodic and chaotic behavior of the chip formation model, which provides an excitation within the machine-tool-workpiece system.

Vibration-assisted machining is another example of externally excited machine tool vibrations, where the small-amplitude, high-frequency tool displacement leads to improved surface finish and accuracy compared to conventional machining [49–52].

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2 The Model and its Linear Stability Analysis

In this paper, a harmonically excited single-degree-of-freedom machine tool vibration model of orthogonal cutting (Figure 1) is investigated (for derivation see [14, 18])

$$\begin{aligned} \ddot{x} + 2\zeta\dot{x} + x = & -p(x - x_\tau) \\ & + q\left((x - x_\tau)^2 - (x - x_\tau)^3\right) \\ & + A\cos(\omega t), \end{aligned} \quad (1)$$

where x is the tool displacement, $\zeta > 0$ is the relative damping factor, $p > 0$ is the nondimensional cutting force, $q > 0$ is the coefficient of nonlinearity, $A \geq 0$ is the amplitude of the forcing and ω is its frequency. The term x_τ denotes $x(t - \tau)$, the delayed value of the position with positive delay τ . The regenerative effect is considered with the expressions containing the dimensionless chip thickness variation $x - x_\tau$ on the right hand side of Eq. (1).

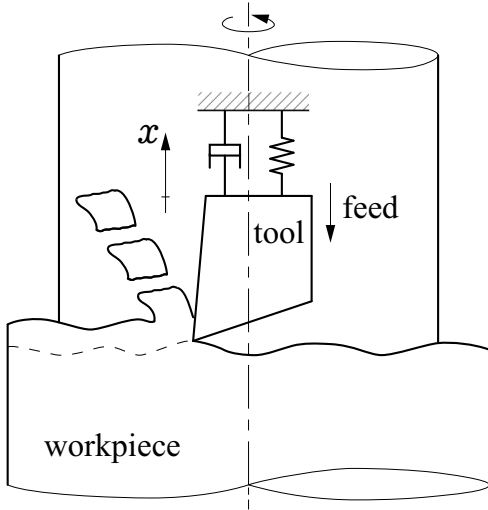


Fig. 1: One-degree-of-freedom mechanical model of orthogonal cutting [19].

The stability analysis of the $x = 0$ solution of the linearized equation

$$\ddot{x} + 2\zeta\dot{x} + x = -p(x - x_\tau), \quad (2)$$

was performed in [36, 53]. The characteristic function of Eq. (2) is obtained by substituting the trial solution $x(t) = C \exp(i\Omega t)$ into Eq. (2). The stability diagram in Figure 2 is given in parametric form (see [14])

$$p = \frac{(1 - \Omega^2)^2 + 4\zeta^2\Omega^2}{2(\Omega^2 - 1)}, \quad (3)$$

$$\tau = \frac{2}{\Omega} \left(j\pi - \arctan \frac{\Omega^2 - 1}{2\zeta\Omega} \right), \quad j = 1, 2, \dots, \quad (4)$$

where j corresponds to the j th ‘lobe’ and $\Omega > 1$. At the minima (‘notches’) of the stability lobes, Ω , p , τ assume the particularly simple forms

$$\Omega_{crit} = \sqrt{1 + 2\zeta}, \quad (5)$$

$$p_{crit} = 2\zeta(\zeta + 1), \quad (6)$$

$$\tau_{crit} = \frac{2 \left(j\pi - \arctan \frac{1}{\sqrt{1+2\zeta}} \right)}{\sqrt{1+2\zeta}}, \quad j = 1, 2, \dots \quad (7)$$

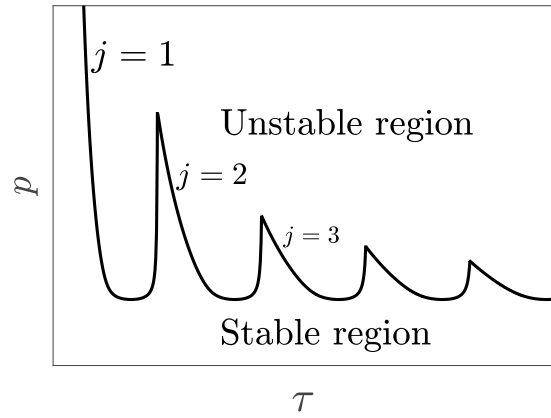


Fig. 2: The linear stability chart of Eq. (1).

Along the stability boundaries, Hopf bifurcation may occur, giving rise to periodic solutions of the nonlinear retarded system [18, 36].

3 Slow Flow Equations and Equilibria

We approximate the solution of Eq. (1) by using the method of multiple scales. In the following we will assume that damping is small, nonlinearity and forcing are weak (see [14]) and the forcing is near-resonant, i.e.

$$\zeta, p, q, A \sim O(\varepsilon), \quad \omega = 1 + \sigma, \quad \sigma \sim O(\varepsilon). \quad (8)$$

Here $\varepsilon \ll 1$ is the bookkeeping parameter and σ is the detuning frequency. We express the system parameters as

$$\begin{aligned} \sigma &= \varepsilon\hat{\sigma}, \quad \omega = 1 + \varepsilon\hat{\sigma}, \\ p &= \varepsilon\hat{p}, \quad q = \varepsilon\hat{q}, \quad \zeta = \varepsilon\hat{\zeta}, \quad A = \varepsilon\hat{A}. \end{aligned} \quad (9)$$

Substituting these into Eq. (1) we get

$$\begin{aligned} \ddot{x} + 2\varepsilon\hat{\zeta}\dot{x} + x = & -\varepsilon\hat{p}(x - x_\tau) \\ & + \varepsilon\hat{q}\left((x - x_\tau)^2 - (x - x_\tau)^3\right) \\ & + \varepsilon\hat{A}\cos((1 + \varepsilon\hat{\sigma})t). \end{aligned} \quad (10)$$

We also assume that the solution of Eq. (10) can be well approximated by the two-scale expansion

$$x(t) = x_0(t_0, t_1) + \varepsilon x_1(t_0, t_1) + O(\varepsilon^2), \quad (11)$$

where the fast and slow timescales are defined as

$$t_0 = t, \quad t_1 = \varepsilon t. \quad (12)$$

With the differential operators

$$D_0 = \frac{\partial}{\partial t_0}, \quad D_1 = \frac{\partial}{\partial t_1}, \quad (13)$$

time differentiation can be written as

$$\frac{d}{dt} = D_0 + \varepsilon D_1 + O(\varepsilon^2), \quad (14)$$

and second derivative with respect to time is

$$\frac{d^2}{dt^2} = D_0^2 + 2\varepsilon D_0 D_1 + O(\varepsilon^2). \quad (15)$$

Substituting the differential operators (14) and (15) into Eq. (10) and equating like powers of ε one obtains

$$\varepsilon^0: \quad D_0^2 x_0(t_0, t_1) + x_0(t_0, t_1) = 0, \quad (16)$$

$$\begin{aligned} \varepsilon^1: \quad D_0^2 x_1(t_0, t_1) + x_1(t_0, t_1) = & -2D_0 D_1 x_0(t_0, t_1) - 2\hat{\zeta} D_0 x_0(t_0, t_1) \\ & -\hat{p}[x_0(t_0, t_1) - x_0(t_0 - \tau, t_1)] \\ & +\hat{q}[x_0(t_0, t_1) - x_0(t_0 - \tau, t_1)]^2 \\ & -\hat{q}[x_0(t_0, t_1) - x_0(t_0 - \tau, t_1)]^3 \\ & +\hat{A}\cos(t_0 + \hat{\sigma}t_1). \end{aligned} \quad (17)$$

Solving Eq. (16) for $x_0(t_0, t_1)$ yields

$$x_0(t_0, t_1) = \alpha(t_1)\cos(t_0 + \beta(t_1)), \quad (18)$$

where $\alpha(t_1)$ and $\beta(t_1)$ are the slowly varying amplitude and phase respectively. We now substitute this solution into Eq. (17). To eliminate the secular terms (terms containing $\sin(t_0 + \beta(t_1))$ and $\cos(t_0 + \beta(t_1))$) we require the following equations to hold:

$$D_1 \alpha = -\left(\hat{\zeta} + \frac{\hat{p}}{2} \sin \tau\right) \alpha - 3\hat{q} \cos \frac{\tau}{2} \sin^3 \frac{\tau}{2} \alpha^3 + \frac{\hat{A}}{2} \sin \phi, \quad (19)$$

$$\alpha D_1 \phi = \frac{1}{2} (\hat{p}(\cos \tau - 1) + 2\hat{\sigma}) \alpha - 3\hat{q} \sin^4 \frac{\tau}{2} \alpha^3 + \frac{\hat{A}}{2} \cos \phi, \quad (20)$$

where we introduced

$$\phi(t_1) = \hat{\sigma}t_1 - \beta(t_1). \quad (21)$$

Eqs. (19, 20) are two ordinary differential equations describing the evolution of the amplitude and phase (slow flow equation). We note that Eq. (20) becomes an algebraic equation for $\alpha = 0$, therefore we only consider the $\alpha > 0$ case.

Without the secular terms, the integration of Eq. (17) gives

$$\begin{aligned} x_1(t_0, t_1) = & \hat{g}_0 \alpha^2(t_1) + \hat{g}_1 \alpha^2(t_1) \cos(2(t_0 + \beta(t_1))) \\ & + \hat{g}_2 \alpha^2(t_1) \sin(2(t_0 + \beta(t_1))) \\ & + \hat{g}_3 \alpha^3(t_1) \cos(3(t_0 + \beta(t_1))) \\ & + \hat{g}_4 \alpha^3(t_1) \sin(3(t_0 + \beta(t_1))), \end{aligned} \quad (22)$$

where

$$\begin{aligned} \hat{g}_0 = & \frac{\hat{q}(1 - \cos(\tau))}{4}, \\ \hat{g}_1 = & \frac{\hat{q}(2\cos(\tau) - \cos(2\tau) - 1)}{6}, \\ \hat{g}_2 = & \frac{\hat{q}(2\cos(\tau) - \cos(2\tau))}{6}, \\ \hat{g}_3 = & \frac{\hat{q}(1 - 3\cos(\tau) + 3\cos(2\tau) - \cos(3\tau))}{32}, \\ \hat{g}_4 = & \frac{\hat{q}(-3\cos(\tau) + 3\cos(2\tau) - \cos(3\tau))}{32}. \end{aligned} \quad (23)$$

Substituting Eq. (18) and Eq. (22) into Eq. (11) and using Eq. (21) we get the multiple scales solution

$$\begin{aligned} x(t) \approx & x_0(t_0, t_1) + \varepsilon x_1(t_0, t_1) \\ = & \alpha(t_1) \cos(\omega t_0 - \phi(t_1)) + g_0 \alpha^2(t_1) \\ & + g_1 \alpha^2(t_1) \cos(2(\omega t_0 - \phi(t_1))) \\ & + g_2 \alpha^2(t_1) (2(\omega t_0 - \phi(t_1))) \\ & + g_3 \alpha^3(t_1) \cos(3(\omega t_0 - \phi(t_1))) \\ & + g_4 \alpha^3(t_1) \sin(3(\omega t_0 - \phi(t_1))), \end{aligned} \quad (24)$$

where $g_i = \varepsilon \hat{g}_i$, $i \in 0, 1, 2, 3, 4$.

3.1 Amplitude of the Steady-State Vibration

The fixed points of the slow flow correspond to periodic solutions of the original equation (1). To get the amplitude α^* and the phase ϕ^* of the steady-state vibration,

i.e. the fixed points of the slow flow equations (19-20), we set the left-hand sides of Eqs. (19-20) to zero; and multiply the expressions by ε , to get back the original ζ, p, q, A, ω system parameters (see Eq. (9))

$$-\left(\zeta + \frac{p}{2} \sin \tau\right) \alpha^* - 3q \cos \frac{\tau}{2} \sin^3 \frac{\tau}{2} \alpha^{*3} + \frac{A}{2} \sin \phi^* = 0, \quad (25)$$

$$\frac{1}{2} (p(\cos \tau - 1) + 2\sigma) \alpha^* - 3q \sin^4 \frac{\tau}{2} \alpha^{*3} + \frac{A}{2} \cos \phi^* = 0. \quad (26)$$

We substitute $\sigma = (\omega - 1)$ and solve the resulting algebraic equations (by eliminating the trigonometric terms from them). From these for the amplitude α^* and for the phase ϕ^* we get

$$\left[\frac{(p(1 - \cos \tau) - 2(\omega - 1) + 6q \sin^4 \frac{\tau}{2} \alpha^{*2})^2}{4} + \left(\zeta + \frac{p \sin \tau}{2} + 3q \cos \frac{\tau}{2} \sin^3 \frac{\tau}{2} \alpha^{*2} \right)^2 \right] \alpha^{*2} - \frac{A^2}{4} = 0, \quad (27)$$

$$\phi^* = \arcsin \left[\frac{2}{A} \left(\left(\zeta + \frac{p}{2} \sin \tau \right) \alpha^* + 3q \cos \frac{\tau}{2} \sin^3 \frac{\tau}{2} \alpha^{*3} \right) \right]. \quad (28)$$

Note that Eq. (27) is an implicit function of the equilibrium amplitude α^* . The shape of the equilibrium amplitude α^* as a function of ω is illustrated in Figure 3 for different A values.

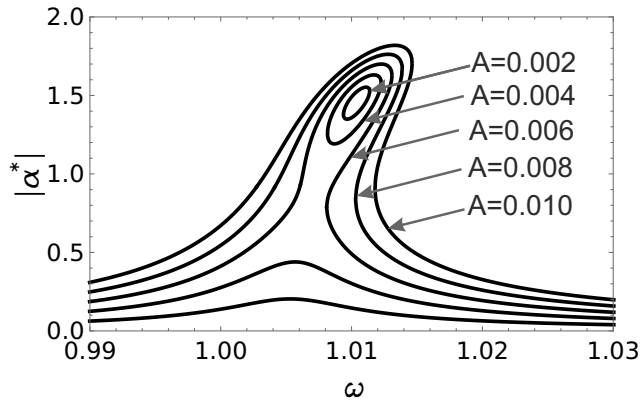


Fig. 3: The fixed point α^* as a function of ω for $A = \{0.002, 0.004, 0.006, 0.008, 0.010\}$. The other parameters are $\tau = 4.676$, $\xi = 0.01$, $p = 0.0101$, $q = 0.003$.

4 Analytical Bifurcation Analysis

4.1 Stability of the Equilibria of the Slow Flow

We examine the stability of the equilibrium points (α^*, ϕ^*) of the slow flow equations (19) and (20). Stability is determined by the eigenvalues

$$\lambda_{1,2} = \frac{\text{tr} \mathbf{J} \pm \sqrt{\text{tr}^2 \mathbf{J} - 4 \det \mathbf{J}}}{2}, \quad (29)$$

where $\mathbf{J}$ is the Jacobian

$$\mathbf{J} = \begin{pmatrix} -\zeta - \frac{p}{2} \sin \tau - 9q \cos \frac{\tau}{2} \sin^3 \frac{\tau}{2} \alpha^{*2} & \frac{A}{2} \cos \phi^* \\ -6q \sin^4 \frac{\tau}{2} \alpha^* - \frac{A}{2\alpha^{*2}} \cos \phi^* & -\frac{A}{2\alpha^*} \sin \phi^* \end{pmatrix}. \quad (30)$$

Using Eq. (25) and (26) we eliminate $\cos \phi^*$ and $\sin \phi^*$ from the Jacobian (30) to yield

$$\mathbf{J} = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}, \quad (31)$$

where

$$\begin{aligned} J_{11} &= -\left(\zeta + \frac{p}{2} \sin \tau\right) - 9q \cos \frac{\tau}{2} \sin^3 \frac{\tau}{2} \alpha^{*2}, \\ J_{12} &= -\frac{1}{2} (p(\cos \tau - 1) + 2\omega - 2) \alpha^* + 3q \sin^4 \frac{\tau}{2} \alpha^{*3}, \\ J_{21} &= \frac{1}{2} (p(\cos \tau - 1) + 2\omega - 2) \frac{1}{\alpha^*} - 9q \sin^4 \frac{\tau}{2} \alpha^*, \\ J_{22} &= -\left(\zeta + \frac{p}{2} \sin \tau\right) - 3q \cos \frac{\tau}{2} \sin^3 \frac{\tau}{2} \alpha^{*2}. \end{aligned} \quad (32)$$

To determine the bifurcation points, the trace and the determinant of the Jacobian $\mathbf{J}$ will be useful

$$\text{tr} \mathbf{J} = -12q \cos \frac{\tau}{2} \sin^3 \frac{\tau}{2} \alpha^{*2} - 2\left(\zeta + \frac{p}{2} \sin \tau\right), \quad (33)$$

$$\det \mathbf{J} = b_2 (\alpha^{*2})^2 + b_1(\omega) (\alpha^{*2}) + b_0(\omega), \quad (34)$$

where

$$\begin{aligned} b_2 &= 27q^2 \sin^6 \frac{\tau}{2}, \\ b_1(\omega) &= 12q \sin^3 \frac{\tau}{2} \left(\zeta \cos \frac{\tau}{2} + (p - \omega + 1) \sin \frac{\tau}{2} \right), \\ b_0(\omega) &= \zeta^2 + \frac{p^2}{2} - p\omega + p + (\omega - 1)^2 + \zeta p \sin \tau \\ &\quad - \frac{1}{2} p(p - 2\omega + 2) \cos \tau. \end{aligned} \quad (35)$$

We note that $\text{tr} \mathbf{J}$ is a linear function and $\det \mathbf{J}$ is a second order polynomial of α^{*2} .

4.2 Saddle-Node Bifurcation

A saddle-node bifurcation of an equilibria occurs when

$$\det \mathbf{J} = b_2 (\alpha^{*2})^2 + b_1(\omega) \alpha^{*2} + b_0(\omega) = 0. \quad (36)$$

Eq. (27) implicitly determines equilibria, and can be written as a third order polynomial of α^{*2}

$$\frac{b_2}{3} (\alpha^{*2})^3 + \frac{b_1(\omega)}{2} (\alpha^{*2})^2 + b_0(\omega) \alpha^{*2} - \frac{A^2}{4} = 0. \quad (37)$$

Eliminating α^{*2} from the simultaneous Eqs. (36) and (37) yields

$$9b_2^2 A^4 + (36b_0(\omega)b_1(\omega)b_2 - 6b_1^3(\omega)) A^2 + 64b_0^3(\omega)b_2 - 12b_0^2(\omega)b_1^2(\omega) = 0. \quad (38)$$

Provided $\frac{16}{3}b_2b_0(\omega) \geq b_1^2(\omega) \geq 4b_2b_0(\omega)$, we get 2 values

$$A = \sqrt{\frac{b_1^3(\omega) - 6b_0(\omega)b_1(\omega)b_2 \pm (b_1^2 - 4b_0(\omega)b_2)^{3/2}}{3b_2^2}}. \quad (39)$$

Eq. (39) determines the two curves on the $\omega - A$ plane where saddle-node bifurcation occurs (Figure 4).

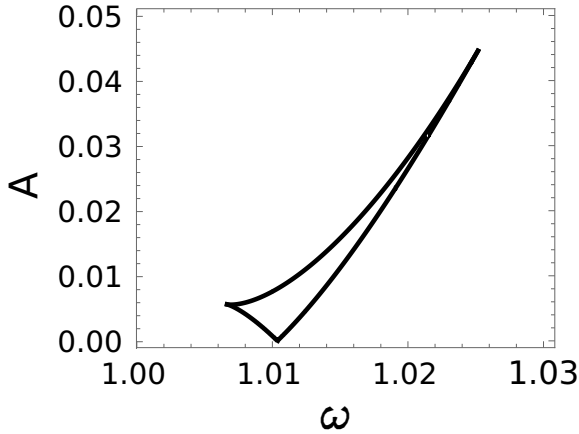


Fig. 4: Saddle-node curves on the $\omega - A$ plane for the parameters $\tau = 4.676$, $\xi = 0.01$, $p = 0.0101$, $q = 0.003$.

4.3 Hopf Bifurcation

Eq. (37) implicitly determines equilibria. A Hopf bifurcation of equilibrium point can occur if $\text{tr} \mathbf{J} = 0$ and $\det \mathbf{J} > 0$ (necessary conditions). To get the sufficient condition of the Hopf bifurcation, the transversality (positive root crossing velocity) and genericity

(equilibrium is weakly attracting/repelling) conditions have to be fulfilled [54].

We start with

$$\text{tr} \mathbf{J} = -12q \cos \frac{\tau}{2} \sin^3 \frac{\tau}{2} \alpha^{*2} - 2(\zeta + \frac{p}{2} \sin \tau) = 0. \quad (40)$$

From Eq. (40) we express

$$\alpha^{*2} = -\frac{\zeta + \frac{p}{2} \sin \tau}{6q \cos \frac{\tau}{2} \sin^3 \frac{\tau}{2}}, \quad (41)$$

and substitute it into Eq. (37) to get an implicit function

$$\mathcal{H}(\omega, A) = h_2 \omega^2 + h_1 \omega + h_0 - \frac{A^2}{4} = 0, \quad (42)$$

where

$$\begin{aligned} h_2 &= -\frac{\zeta + \frac{p}{2} \sin \tau}{6q \cos \frac{\tau}{2} \sin^3 \frac{\tau}{2}} (= \alpha^{*2}), \\ h_1 &= \frac{p^2 + 4 \csc \tau (\zeta \csc^2 \frac{\tau}{2} + (p - \zeta^2) \csc \tau + p \cot \tau)}{6q}, \\ h_0 &= \frac{p^2(p+4) \cos 2\tau - 4\zeta p^2(\sin \tau + \sin 2\tau + 32\zeta^2)}{48q \sin^2 \tau} \\ &\quad - \frac{16\zeta(\zeta^2 + 2 \cos \tau + 2) \csc \tau + 16(\zeta^2 + 1)p \cos \tau}{48q \sin^2 \tau} \\ &\quad + \frac{p^2 + 4p - 8p(\zeta^2 + 2)}{48q \sin^2 \tau}. \end{aligned} \quad (43)$$

We eliminate α^{*2} from the $\det \mathbf{J} > 0$ condition by substituting Eq. (41) into Eq. (34) to get the inequality as the function of ω

$$\begin{aligned} \mathcal{D}(\omega) &= \omega^2 + \left(2\zeta \tan \frac{\tau}{2} - 2\right) \omega + \frac{4\zeta^2 - p^2 + 8}{8(\cos \tau + 1)} \\ &\quad + \frac{(8 - 8\zeta^2 + p^2 \cos \tau) \cos \tau - 4\zeta(p+4) \sin \tau}{8(\cos \tau + 1)} > 0. \end{aligned} \quad (44)$$

Eq. (42) together with the inequality (44) determines the possible Hopf bifurcation points in the (ω, A) plane. For given parameter values such a figure is shown in Figure 12.

When $\mathcal{H}(\omega, A) = 0$ and $\mathcal{D}(\omega) = 0$ we observe Bogdanov-Takens bifurcation points on the Hopf curve.

5 Numerical Results of the Bifurcation Analysis

The angular frequency ω of the forcing was chosen as a bifurcation parameter. As in [18, 55, 56] we set the other parameter values at the first lobe ($j=1$, see Figure 2) and weak nonlinearity and forcing

$$\begin{aligned} \tau &= 4.676, \quad \zeta = 0.01, \quad p = 0.5p_{crit} = 0.0101, \\ q &= 0.003, \quad A = 0.01. \end{aligned} \quad (45)$$

The equilibrium amplitude α^* as a function of ω is illustrated in Figure 5 (this is the uppermost curve of Figure 3).

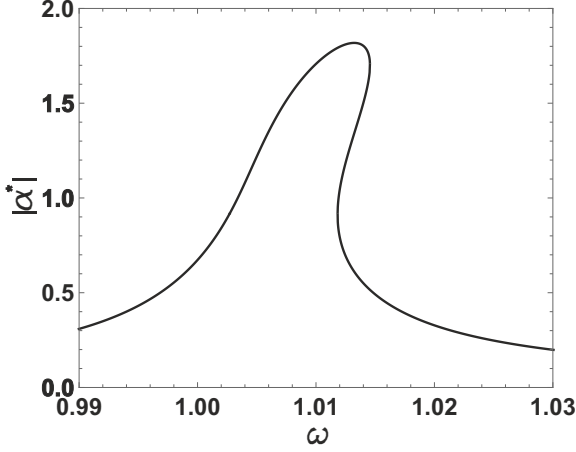


Fig. 5: The equilibrium amplitude as the function of ω .

Using Eq. (39) saddle-node bifurcations occur at

$$\omega_{SN1} = 1.0118, \quad \omega_{SN2} = 1.0146. \quad (46)$$

While at ω_{SN1} a regular saddle-node bifurcation occurs, at ω_{SN2} a so-called homoclinic saddle-node bifurcation [57] can be observed. At this point a limit cycle is born (see Figure 8f and Figure 8g).

Using Eq. (42) and inequality (44), Hopf bifurcation occurs at

$$\omega_{Hopf} = 1.00363. \quad (47)$$

At this point the stable equilibrium point becomes unstable and a subcritical Hopf bifurcation occurs.

The bifurcation diagram is shown in Figure 6.

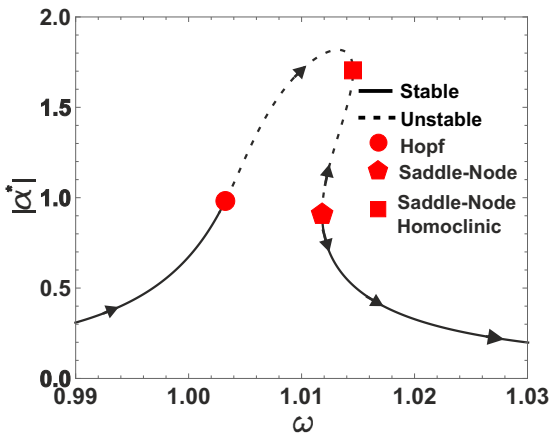


Fig. 6: Bifurcation diagram.

5.1 Phase Plane and Phase Portraits

To illustrate the dynamical behavior of the system, we chose the following angular frequencies of the forcing

$$\begin{aligned} \omega_I &= 1, & \omega_{II} &= \omega_{Hopf} = 1.00363, \\ \omega_{III} &= 1.005, & \omega_{IV} &= \omega_{SN1} = 1.0118, \\ \omega_V &= 1.013, & \omega_{VI} &= \omega_{SN2} = 1.0146, \\ \omega_{VII} &= 1.02. \end{aligned} \quad (48)$$

The right-hand sides of Eqs. (19) and (20) are periodic functions of the phase ϕ with period 2π , thus $(\alpha, \phi) \in (\mathbb{R}^1, S^1)$. The true phase space is a cylindrical surface, see Figure 7.

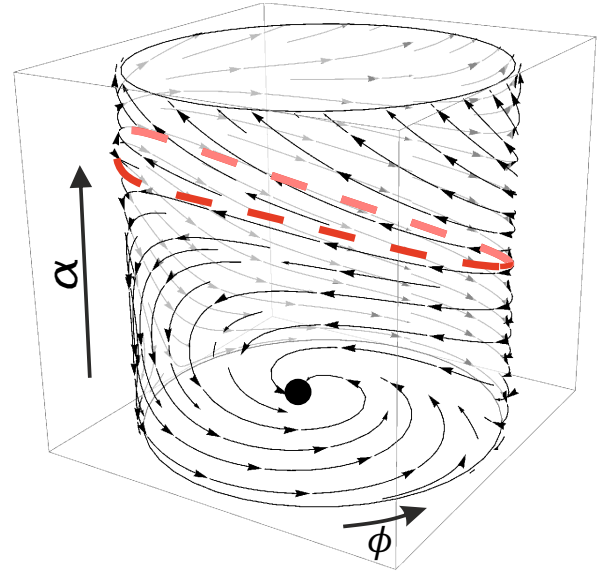


Fig. 7: Phase cylinder at $\omega = \omega_I$. Dashed line - unstable limit cycle, filled circle - stable equilibrium.

Figure 8 shows the phase portrait of the slow flow at various forcing frequencies. The filled circles denote the stable, the empty circles the unstable equilibria and the dashed lines correspond to the unstable limit cycles.

At $\omega = \omega_I$ the equilibrium (filled circle) is a stable spiral and the thick dashed line is a 2π -periodic unstable limit cycle (Figure 8a). At $\omega = \omega_{II}$ a pair of complex conjugate eigenvalues cross the imaginary axis. At this point the fixed point is weakly repelling, giving rise to a subcritical Hopf bifurcation (Figure 8b). After the Hopf bifurcation point ($\omega = \omega_{III}$) the stable equilibrium becomes unstable (Figure 8c) and all solutions go off to infinity. Further increasing the bifurcation parameter ω the unstable spiral equilibrium (empty circle) becomes

an unstable node. In Figure 8d at $\omega = \omega_{IV}$ the left dot ($\alpha^* = 1.72$) is the unstable node, the right dot ($\alpha^* = 0.93$) is a non-hyperbolic fixed point undergoing saddle-node bifurcation. After the saddle-node bifurcation a stable node and a saddle point is created (Figure 8e at $\omega = \omega_V$). The stable node transforms into a stable spiral at $\omega = 1.0119 > \omega_V$ and the saddle point moves towards the unstable node. At $\omega = \omega_{VI}$ the unstable node and saddle point coalesce and a saddle-node homoclinic bifurcation (global bifurcation) occurs (Figure 8f). After the saddle-node homoclinic bifurcation ($\omega = \omega_{VII}$) an unstable limit cycle (thick arrowless line) is created (see Figure 8g).

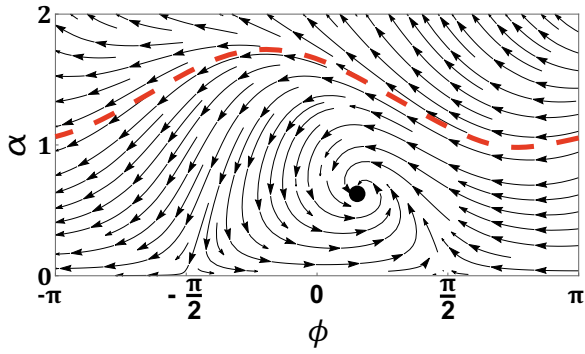
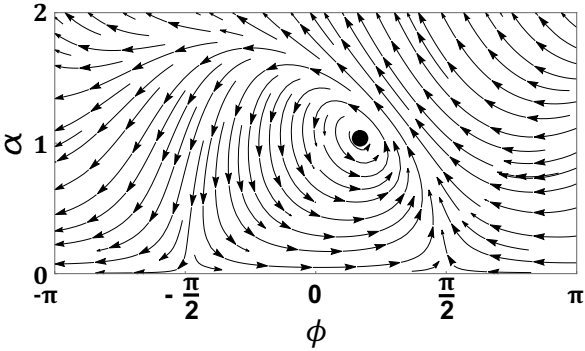
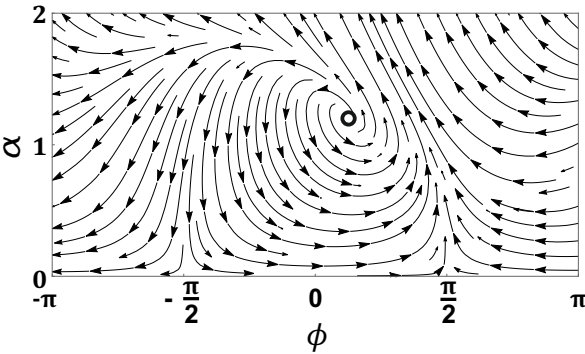
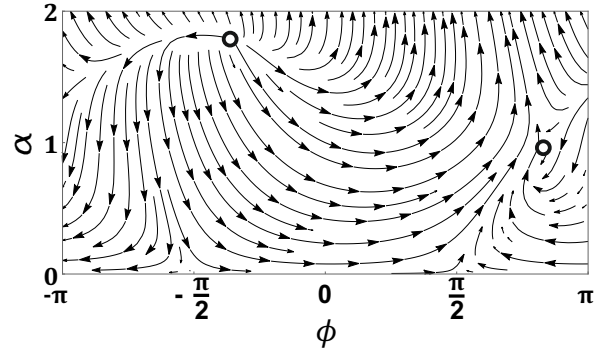
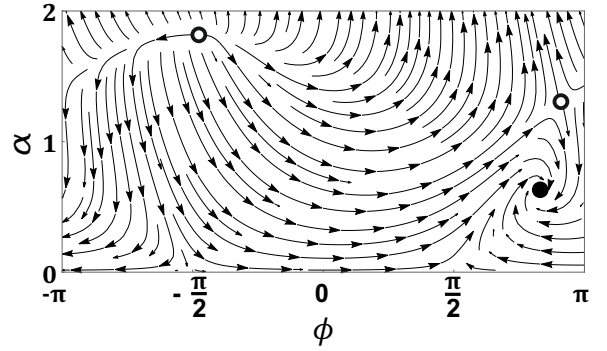
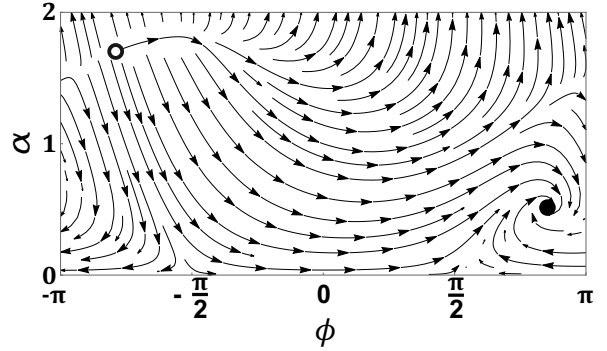
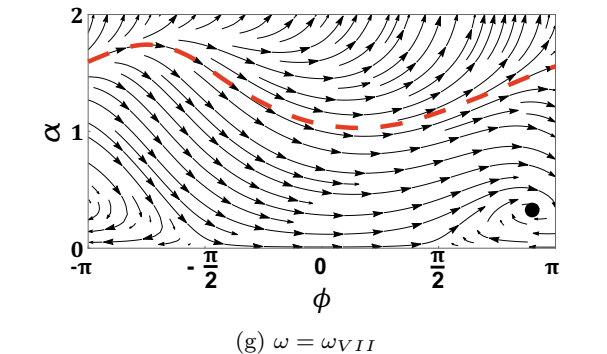
(a) $\omega = \omega_I$ (b) $\omega = \omega_{II}$ (c) $\omega = \omega_{III}$ (d) $\omega = \omega_{IV}$ (e) $\omega = \omega_V$ (f) $\omega = \omega_{VI}$ (g) $\omega = \omega_{VII}$

Fig. 8: Phase portrait at various forcing frequencies.

5.2 Global Bifurcations

At $\omega = 1.00327 < \omega_{Hopf}$ the unstable limit cycle around the equilibrium point “collides” with the non-admissible line $\alpha = 0$ giving rise to a “global” (2π -periodic in ϕ) unstable limit cycle (Figure 9).

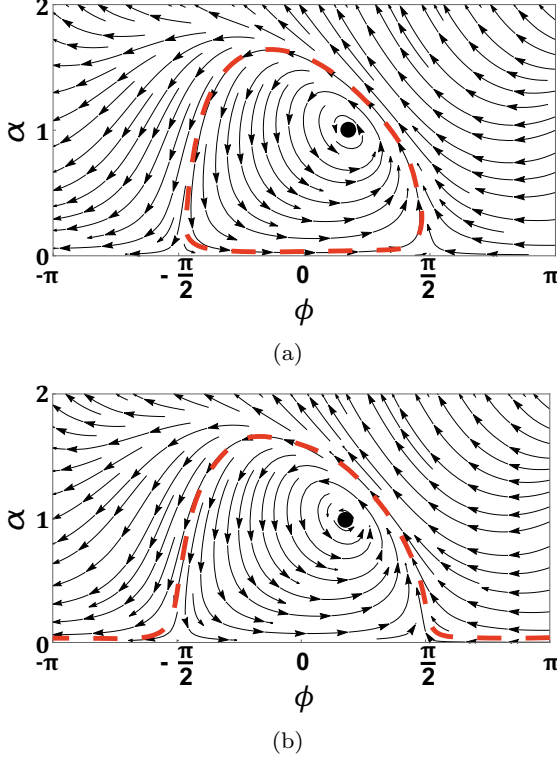


Fig. 9: Trajectories before and after the global bifurcation.

Another global bifurcation is a homoclinic saddle-node bifurcation [57]. This occurs at $\omega = \omega_{VI}$. The coalescence of the saddle and unstable node is a homoclinic bifurcation and a “global” unstable limit cycle is born (see Figure 10).

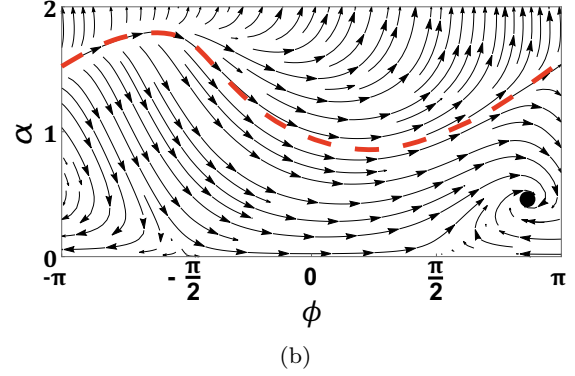


Fig. 10: Trajectories before and after the homoclinic saddle-node bifurcation.

5.3 The $\omega - A$ plane

Now we consider the forcing frequency ω and amplitude A as two bifurcation parameters. The other system parameters were the same as in Eq. (45). We determined the saddle-node curves together with the cusp and Bogdanov-Takens points with MatCont [58–61] (see Figure 11). The boundary of the gray region is the same as the analytical result (39) (see also Figure 4).

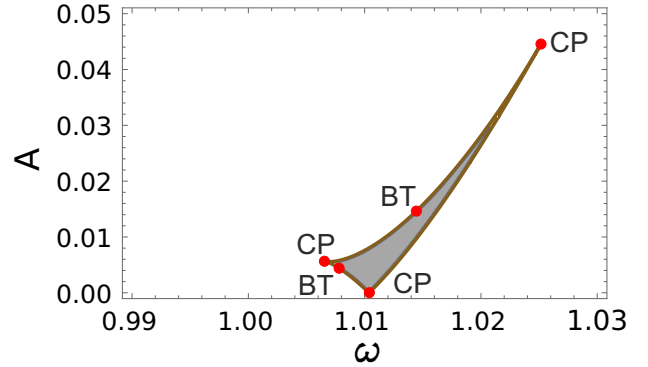


Fig. 11: Bifurcation diagram (CP - cusp, BT - Bogdanov-Takens).

Inside the closed wedge (filled with gray in Figure 11) three equilibrium points of the slow flow exist, two on the boundary and one outside the wedge. At the corners of the wedge we have three cusp (CP) bifurcation points. Eqs. (42) and (44) determine two Bogdanov-Takens (BT) points [54]. Figure 12 shows the Hopf bifurcation curve determined by Eq. (42).

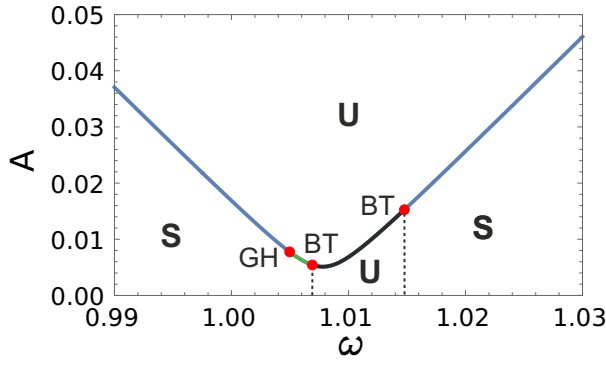


Fig. 12: Bifurcation diagram (BT - Bogdanov-Takens, GH - Generalized Hopf, **S** - stable region, **U** - unstable region).

Using MatCont we determined the Generalized Hopf (Bautin bifurcation) point. At the Generalized Hopf bifurcation point the first Lyapunov coefficient vanishes [54]. This bifurcation point separates branches of subcritical and supercritical Hopf bifurcations (left segment until GH and GH-BT segment in Figure 12). The segment BT-BT corresponds to neutral saddle points. In a neighborhood of the GH point, fold (saddle-node) bifurcation of limit cycles occur. To illustrate this, we chose 4 points (denoted by squares in Figure 13), one before and three after the Generalized Hopf bifurcation point.

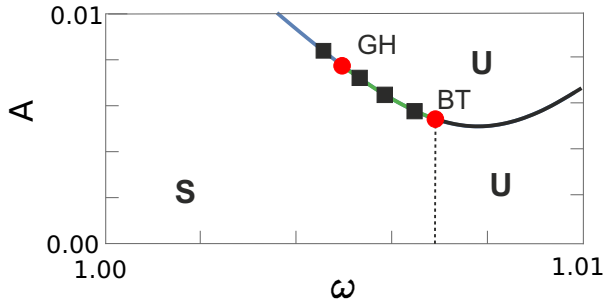
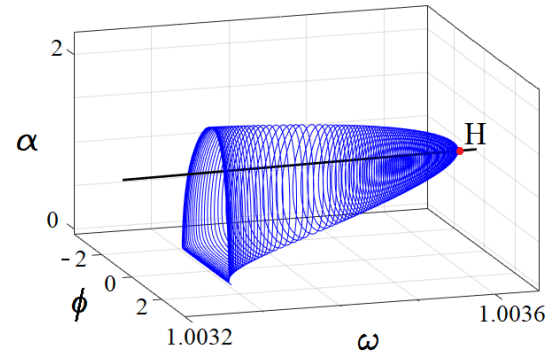
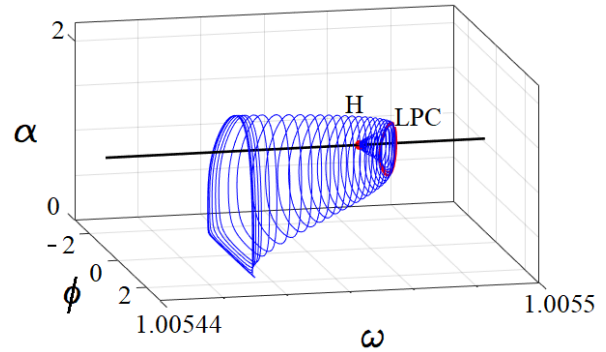


Fig. 13: Bifurcation diagram (BT - Bogdanov-Takens, GH - Generalized Hopf, **S** - stable region, **U** - unstable region). The squares denote the starting points of the limit cycle continuation.

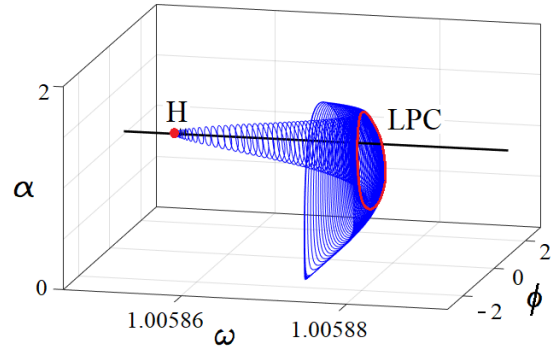
To trace the limit cycles we used MatCont, with the bifurcation parameter ω . At the first point $A = 0.0075$ the Hopf bifurcation is subcritical (Figure 14a). After the Generalized Hopf bifurcation point the Hopf bifurcation becomes supercritical. At $A = 0.0070$ and $A = 0.0065$ the system has two limit cycles which collide and disappear in a fold bifurcation (Figure 14b and 14c).



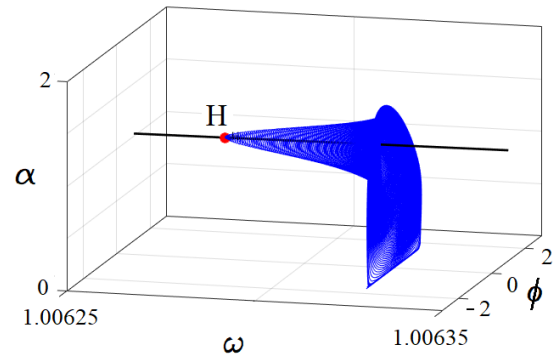
(a) $A = 0.0075$



(b) $A = 0.0070$



(c) $A = 0.0065$



(d) $A = 0.0060$

Fig. 14: The limit cycles before and after the General Hopf bifurcation point (H - Hopf bifurcation point, LPC - Limit Point bifurcation of Cycles, the MatCont terminology for fold bifurcation of limit cycles).

The two coexisting limit cycles (small limit cycle - stable, large limit cycle - unstable) at $\omega = 1.00587$, $A = 0.0065$ is illustrated in Figure 15. At $A = 0.0060$ the system has one limit cycle again (Figure 14d).

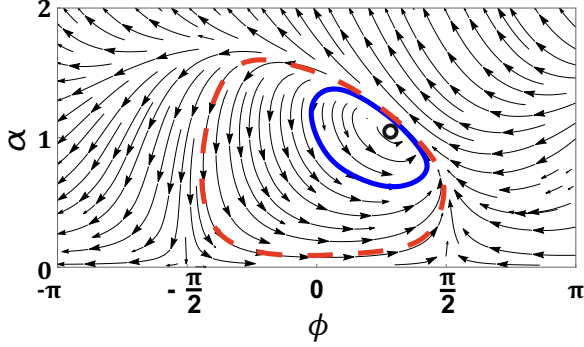


Fig. 15: Phase plane at $\omega = 1.00587$, $A = 0.0065$, with 2 coexisting limit cycles (solid line - stable, dashed line - unstable).

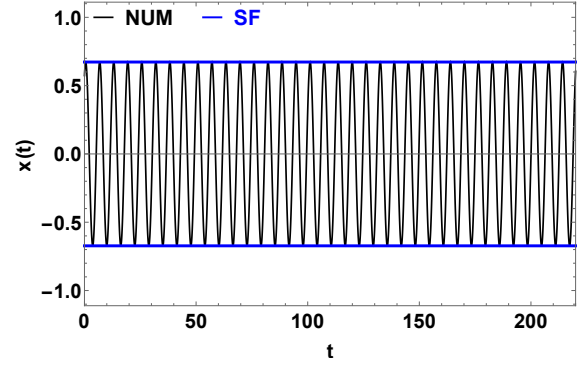
6 Comparison of the Method of Multiple Scales and Direct Numerical Solution

The direct numerical solution of Eq. (1) with parameters specified in Eq. (45) was determined in Mathematica using the built in adaptive Runge-Kutta method. The numerical simulation is started with the initial function

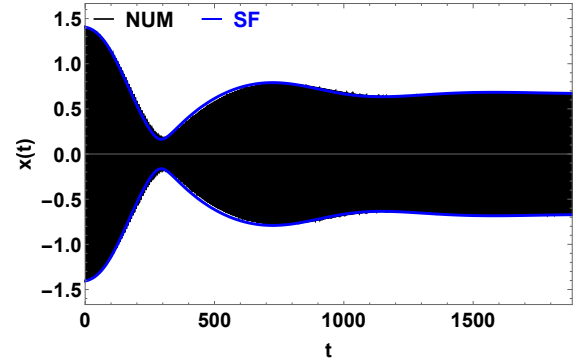
$$\begin{aligned} x(t) = & \alpha^* \cos(\omega t - \phi^*) + g_0 \alpha^{*2} \\ & + g_1 \alpha^{*2} \cos(2(\omega t - \phi^*)) + g_2 \alpha^{*2} (2(\omega t - \phi^*)) \\ & + g_3 \alpha^{*3} \cos(3(\omega t - \phi^*)) \\ & + g_4 \alpha^{*3} \sin(3(\omega t - \phi^*)), \quad t \in [-\tau, 0]. \end{aligned} \quad (49)$$

where α^* and ϕ^* were determined numerically from Eq. (27) and (28).

Figures 16 illustrates the direct numerical solution (NUM) of Eq. (1) and the slow flow (SF) solution Eqs. (19) and (20). The fixed points of the slow flow equations (constant amplitude $x(t)$ in Figure 16a) are the periodic solution of the original delay equation. Figure 16b shows that the slow flow equations capture the amplitude dynamics even away from an equilibrium point.



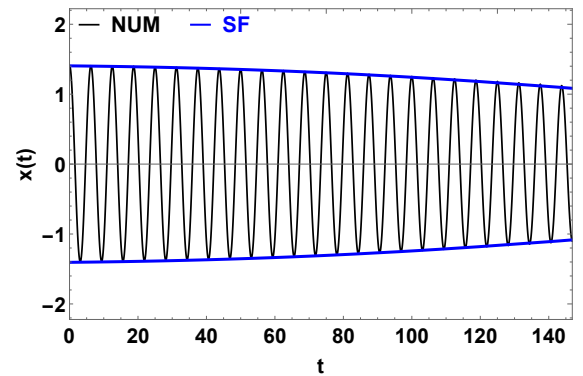
(a) Steady-state vibration.



(b) Transition from transient response to steady-state vibration.

Fig. 16: Method of multiple scales solution and direct numerical solution ($A = 0.01$, $\omega = 1$).

The unstable limit cycles of the subcritical Hopf bifurcations of the slow flow equations (19, 20) are unstable quasi-periodic solutions of Eq. (1). Figure 17 presents the direct numerical solution (NUM) of Eq. (1) and the slow flow (SF) solution Eq. (19). Figure 17a demonstrates that initially the slow flow solution correctly approximates the envelope of the numerical solution, while Figure 17b shows the long-term drift of the phase.



(a) $t \in [0, 30\tau]$

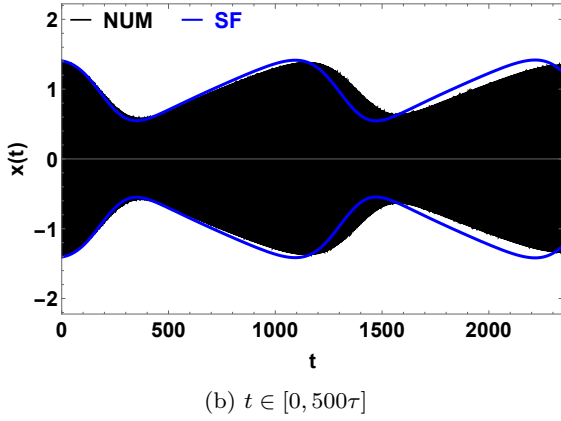


Fig. 17: Method of multiple scales solution and direct numerical solution ($A = 0.01$, $\omega = 1.0034$) shown on different time intervals.

6.1 Primary Resonance

The amplitude of the first harmonic (with frequency ω) of the numerical solution was determined in the time interval $[T_1, T_2] = [10\tau, 50\tau]$, with the following expression

$$\alpha_{NUM}^* = \sqrt{\alpha_{\sin}^2 + \alpha_{\cos}^2}, \quad (50)$$

where

$$\alpha_{\sin} = \frac{2}{T_2 - T_1} \int_{T_1}^{T_2} x(t) \sin(\omega t) dt, \quad (51)$$

$$\alpha_{\cos} = \frac{2}{T_2 - T_1} \int_{T_1}^{T_2} x(t) \cos(\omega t) dt. \quad (52)$$

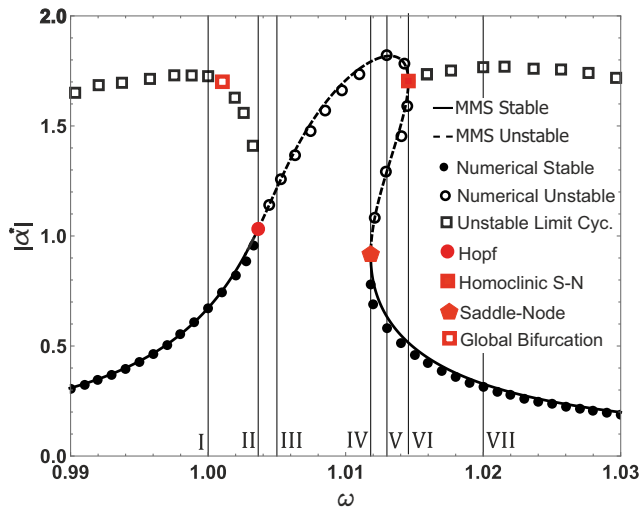


Fig. 18: Primary resonance curve.

Figure 18 shows the amplitude α^* (Eq. (27)) as a function of the forcing frequency ω . The solid line indicates the stable multiple scales solution (MMS Stable), the dashed line shows the unstable multiple scales solution (MMS Unstable). The stable numerical equilibrium solutions are shown with filled circles, unstable numerical solution are depicted by empty circles. The solution was deemed unstable if its magnitude grew beyond a large number. The amplitude of the unstable limit cycles are depicted by empty squares (these amplitudes were determined using MatCont and bisection method). The thin vertical lines indicate the ω_i , $i \in \{I, II, \dots, VII\}$ angular frequencies. The red markers correspond to the subcritical Hopf (Hopf), homoclinic saddle-node (Homoclinic S-N), global bifurcation of limit cycles (Global) and saddle-node (Saddle-Node) bifurcation points.

7 Conclusions

Eq. (1) without forcing admits a subcritical Hopf bifurcation [18]. Forcing and Hopf bifurcation together may yield complicated motions [62]. Plaut and Hsieh [63] studied a forced 1-DOF mechanical system with delay and found periodic, chaotic and unbounded responses. Daqaq et al. investigated a harmonically forced delay system and got similar primary resonance curve [23]. The Taylor-expansion of forced delay equation of machine tool vibration was investigated in [10].

Here we utilized the method of multiple scales to derive the slow flow equations for the forced delay equation Eq. (1). Equilibria of the slow flow were given by implicit algebraic equations. Analytical expressions were derived for the saddle-node and Hopf bifurcations of the slow flow equations. Bifurcation analysis of the slow flow equations has been shown for some numerical values of the parameters. Sub- and supercritical Hopf, generalized Hopf (Bautin), saddle-node, homoclinic saddle-node bifurcations were found. The bifurcations were illustrated with phase portraits and direct numerical integration of the original equation. Using MatCont we determined the Generalized Hopf point of the system and located the local limit cycles. The analysis presented here demonstrates the rich dynamics of the system.

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Compliance with ethical standards

Conflicts of interest

The authors declare that they have no conflict of interest.

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