

Linearly conjugate realization of uncertain kinetic systems with time delay

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1 Introduction

Delayed systems are ubiquitous in nature and technology, and it is often required to explicitly include time delay into dynamical models to describe the studied processes in a satisfactory way [1]. The original motivation of the application of delayed models in chemistry was to avoid the detailed modeling of intermediates and thus to concentrate on the description of key species [2]. However, higher level models that are often formally kinetic such as compartmental or process models, may contain inherent (e.g., transport or communication) delays which can be modeled as delayed reactions.

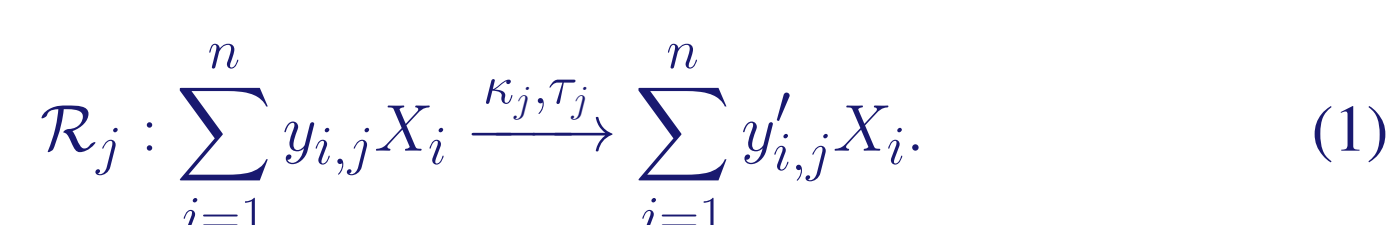
The problem of kinetic realization, namely, how to algorithmically construct a reaction network with a prescribed nonnegative dynamics given in the form of polynomial ordinary differential equations was first addressed and solved in [3]. It was immediately visible from the solution that the reaction graph of a given set of kinetic ODEs is generally non-unique. This non-uniqueness was used in an optimization framework to compute preferred or even all possible reaction graphs realizing a polynomial dynamics (see, e.g. [4] and the related articles).

The purpose of this work is to propose a solution for the kinetic realization problem of nonnegative polynomial delayed differential equations (DDEs) by using and extending the above mentioned results.

2 Mathematical background

2.1 Mass-action reaction networks with time delays

Delayed CRNs consist of elementary reaction steps $\mathcal{R}_j$, $j = 1, \dots, r$, taking place between n chemical species denoted by $X_1, \dots, X_n$ of the form



where

- $y_{i,j}, y'_{i,j} \in \mathbb{N}$ are **stoichiometric coefficients**,
- $\kappa_j \in \mathbb{R}^+$ are **reaction rate coefficients**,
- $\tau_j \in \mathbb{R}_0^+$ are **time delays**.

The linear combinations of species on the left and right hand sides of the reaction steps (1) are called **complexes** (source complex and product complex, respectively)

$$C_j = \sum_{i=1}^n y_{i,j} X_i \quad C'_j = \sum_{i=1}^n y'_{i,j} X_i. \quad (2)$$

Note that the zero complex can also be present in the set of complexes, we can use it to represent the effects of the environment. The mass action law states that the reaction rate of the reaction step (1) is

$$\varrho_j(x) = \kappa_j \prod_{i=1}^n x_i^{y_{i,j}} \quad (3)$$

where x_i denotes the concentration of the specie X_i .

2.2 Graph representation of mass-action systems with time delays

We can assign a weighted directed graph

$D = (V, E) = (V, E^{T_0=0}, E^{\tau_1}, \dots, E^{\tau_r})$ to the reaction network (1)

1. the vertices correspond to the complexes, i.e., $V = \{C_1, \dots, C_m\}$,
2. the directed edges represent reactions, i.e., $(C_j, C_l) \in E^{\tau_i}$, if C_j transforms into C_l in the network with time delay τ_i ,
3. the reaction rate coefficients are assigned as weights to the edges.

2.3 Delay differential equations of mass-action systems with time delays

The dynamics of a delayed CRN with reaction steps (1) is described by a system of delay differential equations (DDEs) of the form

$$\dot{x}(t) = \sum_{j=1}^r \kappa_j [(x(t - \tau_j))^{y_j} y'_j - (x(t))^{y_j} y_j] \quad (4)$$

where $x \in \mathbb{R}^n$ is the concentration vector of the species. It can be shown that we can describe the dynamics in the equivalent form

$$\dot{x}(t) = \sum_{i=0}^p Y A_i \psi(x(t - \tau_i)) \quad (5)$$

- $Y \in \mathbb{R}^{n \times m}$ is the **stoichiometric composition matrix**,
- $A_i \in \mathbb{R}^{m \times m}$, $i = 0, \dots, p$ contain the reaction rate coefficients,
- $\psi : \mathbb{R}^n \mapsto \mathbb{R}^m$ is the **monomial mapping** defined by

$$\psi_j(x) = \prod_{i=1}^n x_i^{y_{i,j}}, \quad j = 1, \dots, m. \quad (6)$$

The exact structure of A_i is

$$[A_0]_{j,k} = \begin{cases} -\sum_{i=0}^p \sum_{l=1}^m \kappa_{j,l}^{\tau_i}, & j = k \\ \kappa_{k,j}^{\tau_0}, & j \neq k \end{cases} \quad (7)$$

$$[A_i]_{j,k} = \kappa_{k,j}^{\tau_i} \quad i = 1, \dots, p. \quad (8)$$

where $\kappa_{k,j}^{\tau_i}$ is the weight of the $(C_k, C_j) \in E^{\tau_i}$ edge.

Note that if $\tau_i = 0$, $i = 1, \dots, p$, then the above system reduces to a system of ODEs describing an ordinary CRN with $A = \sum_{i=0}^p A_i$ containing the weighted directed edges.

The A_i matrices characterize the tensor

$$\mathcal{A}^p = [A_0 \dots A_p] \in \mathbb{R}^{(p+1) \times m \times m}. \quad (9)$$

The entries of $\mathcal{A}^p$ are used to build the column vector

$$\text{vec}(\mathcal{A}^p) = \left[[A_0]_{j \neq 1,1}^T \dots [A_0]_{j \neq m,m}^T \dots [A_p]_{j \neq 1,1}^T \dots [A_p]_{j \neq m,m}^T \right]^T \quad (10)$$

3 Uncertain kinetic models with time delay

Let us consider a dynamical system with time delays

$$\dot{x}(t) = \sum_{i=0}^p M_i \psi(x(t - \tau_i)). \quad (11)$$

The M_i matrices characterize the tensor

$$\mathcal{M}^p = [M_0 \dots M_p] \in \mathbb{R}^{(p+1) \times n \times m} \quad (12)$$

which, along with the complex composition matrix Y uniquely describe the system. The entries of $\mathcal{M}^p$ are used to build the column vector

$$\text{vec}(\mathcal{M}^p) = \left[[M_0]_{\cdot,1}^T \dots [M_0]_{\cdot,m}^T \dots [M_p]_{\cdot,1}^T \dots [M_p]_{\cdot,m}^T \right]^T. \quad (13)$$

The uncertainty will be modelled by assuming that the $\mathcal{M}^p$ tensor is

- constant, but **unknown**,
- represented by a point of a $(p+1)nm$ dimensional **polytope**

$$\mathcal{P} = \{q \in \mathbb{R}^{(p+1)nm} \mid n_i^T q \leq b_i, i = 1, \dots, l\}. \quad (14)$$

$[\mathcal{P}, Y]$ represents a **delayed uncertain kinetic system**.

Definition 1. A delayed reaction network $(Y, \mathcal{A}^p)$ is called a **linearly conjugate realization** of the delayed uncertain kinetic system $[\mathcal{P}, Y]$ if

- there exists $\mathcal{M}^p \in \mathbb{R}^{(p+1) \times n \times m}$ and $T \in \mathbb{R}^{n \times n}$ **positive definite diagonal matrix**, such that

$$Y \mathcal{A}^p \underbrace{\text{diag}(\psi(T \cdot \mathbf{1}))^{-1}}_{\Psi_T^{-1}} = T^{-1} \mathcal{M}^p,$$

- $\text{vec}(T^{-1} \mathcal{M}^p) \in \mathcal{P}$.

We will refer to this realization as the triple $(T^{-1}, \mathcal{M}^p, \mathcal{A}^p \Psi_T^{-1})$.

We also consider a set L of finitely many **additional linear constraints** affecting the entries of T^{-1} and $\mathcal{A}^p \Psi_T^{-1}$ **describing desired special properties** of the realizations, e.g. weak reversibility. $[\mathcal{P}, L, Y]$ represents a **constrained delayed uncertain kinetic system**.

Definition 2. A delayed reaction network $(Y, \mathcal{A}^p)$ is called a **linearly conjugate realization** of the constrained delayed uncertain kinetic system $[\mathcal{P}, L, Y]$ if

- it is a linearly conjugate realization of $[\mathcal{P}, Y]$,
- the entries of T^{-1} and $\mathcal{A}^p \Psi_T^{-1}$ fulfil the set L of constraints.

We will refer to this realization as the triple $(T^{-1}, \mathcal{M}^p, \mathcal{A}^p \Psi_T^{-1})$.

The following definitions and propositions will help us understand the exact structure of the linearly conjugate realizations of a constrained delayed uncertain kinetic system.

Definition 3. A linearly conjugate realization of a constrained delayed uncertain kinetic system

1. is **sparse**, if it contains **minimal number of reactions**,
2. is **dense**, if it contains **maximal number of reactions**,
3. has **superstructure property**, if its reaction graph contains the reaction graph of all other linearly conjugate realizations of the delayed kinetic system as subgraphs.

Using the above we can now state the following very important propositions.

Proposition 1. The linearly conjugate realizations of a constrained delayed uncertain kinetic system form a convex set.

Proposition 2. The dense linearly conjugate realization of a constrained delayed uncertain kinetic system has superstructure property.

Corollary 1. The structure of the unweighted directed reaction graph of the dense linearly conjugate realization of a constrained delayed uncertain kinetic system is unique.

Note that these general results enable us to compute a wide variety of structures as **special cases**:

- **dynamical equivalence** is the special case of linear conjugacy when $T, T^{-1}, \Psi_T, \Psi_T^{-1}$ are **identity matrices**,

- **non-delayed** system are special cases where **each delay is zero**,

- **non-uncertainty** is a special case when $\mathcal{P}$ is a **single point** in the parameter space.

4 Example: delayed Oregonator model

Let us consider the dimensionless form of the Oregonator model with time delays [2]

$$\begin{aligned} \dot{x}(\theta) &= \alpha(f\beta y(\theta) - f\alpha x(\theta)y(\theta) + \alpha x(\theta) - 2\beta x^2(\theta)) \\ \dot{y}(\theta) &= -y(\theta) - \frac{\alpha}{\beta} x(\theta)y(\theta) + \frac{\alpha}{\beta} x(\theta - \epsilon). \end{aligned}$$

We will model uncertainty by assuming that the above parameters are in the cuboid defined by two classical parametrizations of the Oregonator model, namely the FKN and Field-Försterling parameter sets.

4.1 Computation of reaction graph structures

Figure 1 shows the reaction graphs of the sparse realizations.

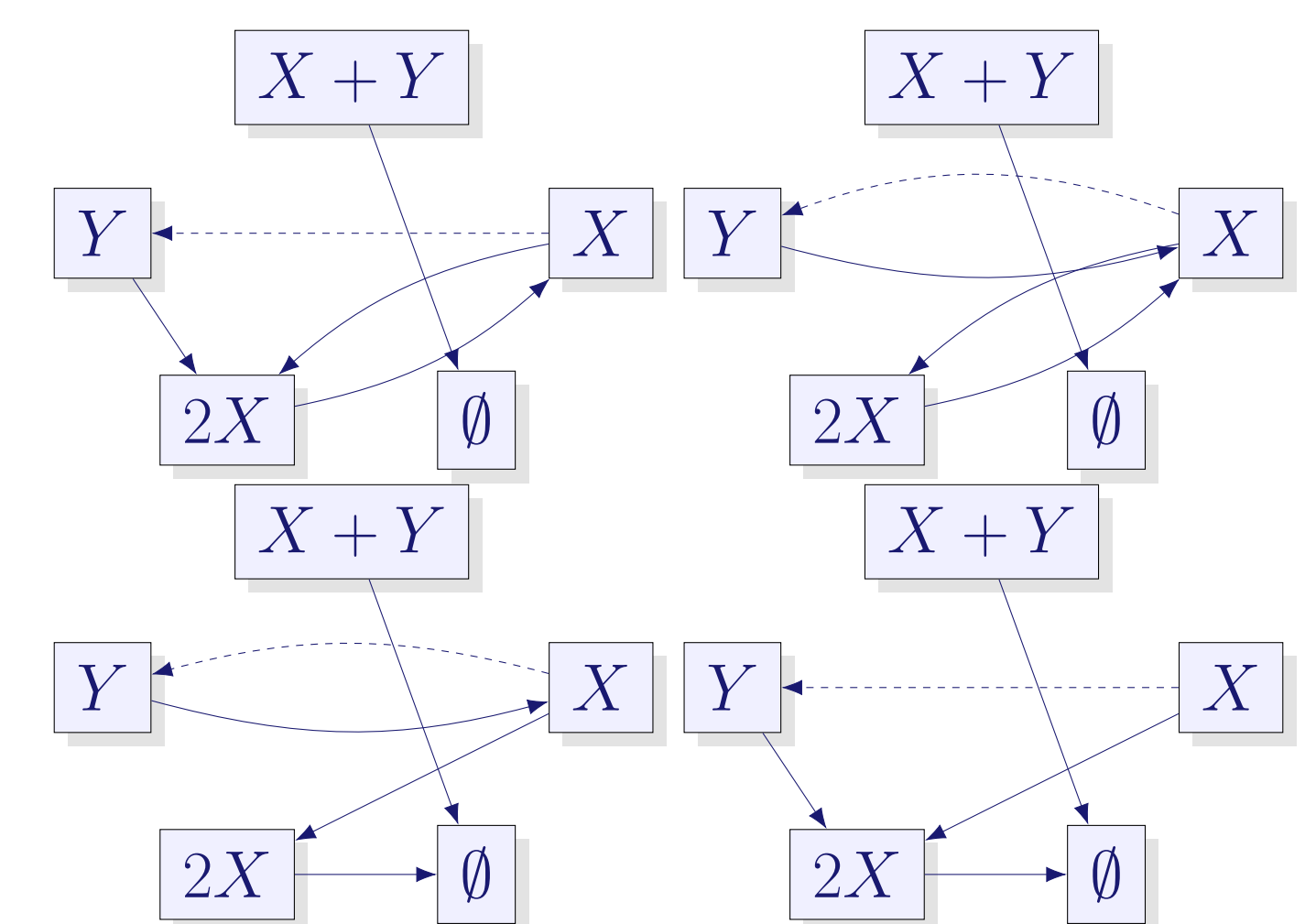


Figure 1: Reaction graphs of sparse realizations

Figure 2 shows the distribution of structurally different realizations depending on the number of reactions.

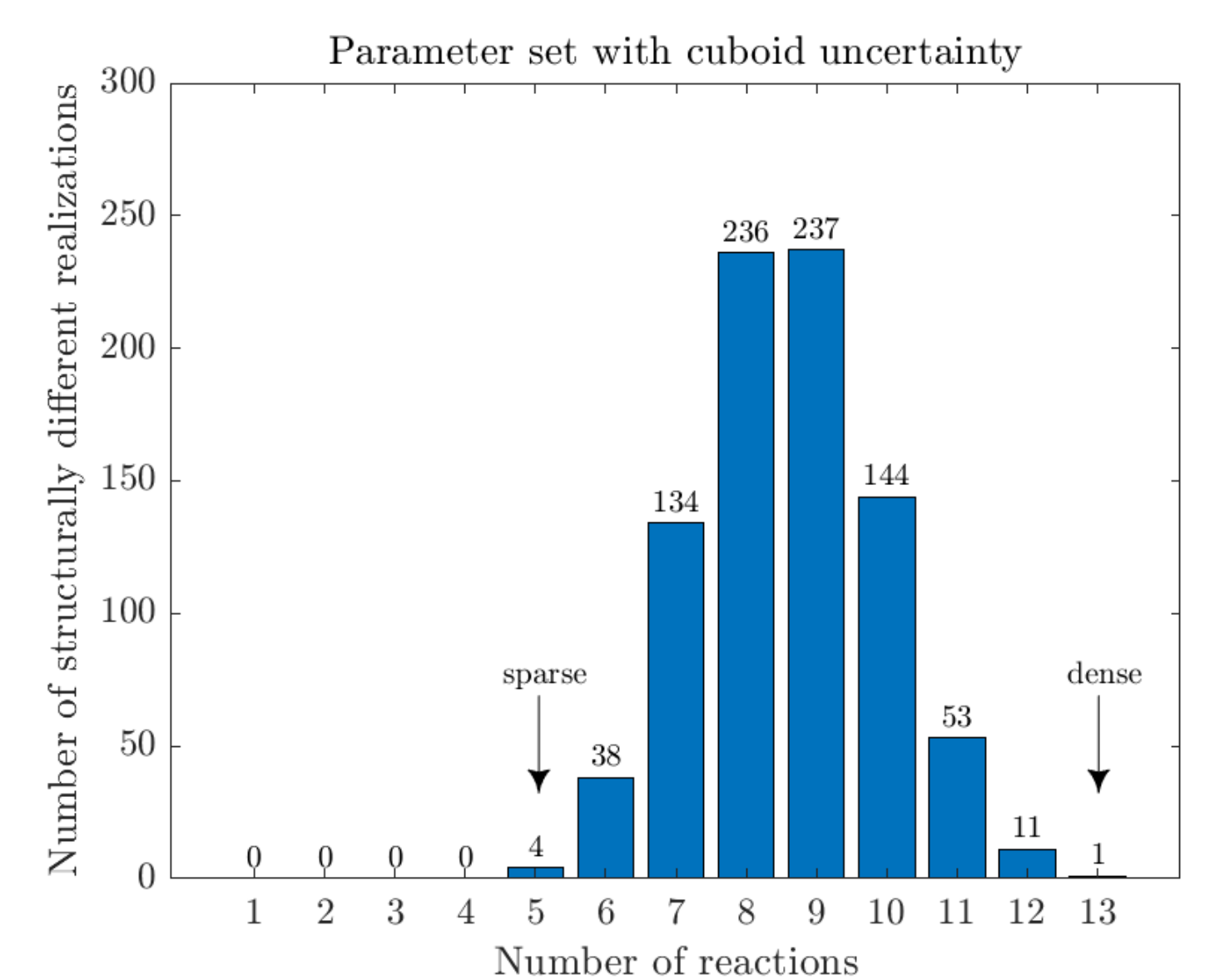


Figure 2: Distribution of structurally different realizations

5 Conclusion

In this work we

- proved that linearly conjugate **realizations form a convex set**,
- proved that **dense realization has superstructure property**,
- generalized optimization model and algorithms to compute (i) the **dense realization**, (ii) **all realizations** and (iii) **weakly reversible realizations**.

Acknowledgements



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