

Connectedness of the Free Uniform Spanning Forest as a function of edge weights

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Abstract

Let G be the Cartesian product of a regular tree T and a finite connected transitive graph H . It is shown in [3] that the Free Uniform Spanning Forest (FSF) of this graph may not be connected, but the dependence of this connectedness on H remains somewhat mysterious. We study the case when a positive weight w is put on the edges of the H -copies in G , and conjecture that the connectedness of the FSF exhibits a phase transition. For large enough w we show that the FSF is connected, while for a large family of H and T , the FSF is disconnected when w is small (relying on [3]). Finally, we prove that when H is the graph of one edge, then for any w , the FSF is a single tree, and we give an explicit formula for the distribution of the distance between two points within the tree.

1 Introduction

Consider some finite graph H with a weight function (“conductances”) $\hat{w} : E(H) \rightarrow \mathbb{R}_0^+$ on its edges. One may take an unweighted graph and view it as one where the weights are constant 1. Choose a spanning tree of H at random, where the probability of a spanning tree T will be proportional to $\prod_{e \in E(T)} \hat{w}(e)$. The so-defined probability measure is called the *Uniform Spanning Tree* (UST) of $(H, \hat{w})$. For a given *infinite* graph G and conductances $\hat{w}$, consider some exhaustion of G by a sequence of connected finite graphs G_n , and let $\text{UST}(G_n)$ be the UST of the weighted graph $(G_n, \hat{w}|_{G_n})$. It is known that the weak limit of $\text{UST}(G_n)$ exists, meaning that for any $e_1, \dots, e_k, f_1, \dots, f_m \in E(G)$, $\mathbb{P}(e_1, \dots, e_k \in \text{UST}(G_n), f_1, \dots, f_m \notin \text{UST}(G_n))$ converges, and to the same limit for any choice of the sequence G_n . The limiting measure is called the *Free Uniform Spanning Forest* (FUSF or FSF) of $(G, \hat{w})$. See [2] for background, references, and the basic properties of the FSF.

It was generally expected that the FSF of “tree-like graphs” would consist of a single tree, until Gábor Pete and the last author showed in [3] that for suitably chosen d and connected finite transitive graph H , the Cartesian product $\mathbb{T}^d \square H$ of the d -regular tree $\mathbb{T}^d$ and H has a disconnected FSF. From the proof, however, it is not clear what happens to the disconnectedness of the FSF if we do some natural changes to d and H , e.g., increase d with a fixed H , or fix d and take a lift of H . No monotonicity result of this type is known. A question in the same spirit is to ask how the connectedness of the FSF changes if we put constant positive weight w on every edge of the H -copies in $\mathbb{T}^d \square H$ and then change w . What happens if w is very small or large? Does there always exist some w where there is a single FSF component? Is there always some w where there are infinitely many FSF components (with an H that has at least 2 vertices)? Is there any kind of monotonicity in w , and perhaps even a critical value that separates the phases of disconnectedness and connectedness? The present paper contributes to the understanding of these questions. In particular, initial steps are taken in Conjecture 1.1.

Let H be a finite connected graph and $\mathbb{T}^d$ be the d -regular tree. For an arbitrary given $w > 0$, define the weight function $\hat{w}$ on the edges of $G = \mathbb{T}^d \square H$ so that $\hat{w}(e) = 1$ if e is of the form $\{(x_1, y), (x_2, y)\}$ and $\hat{w}(e) = w$ if e is of the form $\{(x, y_1), (x, y_2)\}$. Define $\text{FSF}_w(G)$ as the FSF of $(G, \hat{w})$.