

Continuous-time approach of sampled-data systems

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Abstract

This paper studies the continuous-time behaviour of sampled-data control systems. In these systems, a discrete-time control law is applied to a continuous-time system model using sampling and zero-order hold. Traditionally, sampled-data systems are modeled in discrete-time, and thus the intersample dynamics cannot be examined. This paper shows that the classical methods (such as the *d2c* function of MATLAB) for obtaining continuous-time models can be inaccurate between the sampling instants, even when Shannon's sampling theorem is satisfied. In this paper, a new method is also presented which can be used to characterize the continuous-time dynamics of sampled-data control systems. The new continuous-time system model including the intersample behavior is derived analytically from the discrete-time model, and the results are verified by simulations.

Keywords

sampling, closed-loop control, sampled-data systems, intersample dynamics

1 Introduction

Sampled-data systems play a fundamental role in mechatronics due to the digital realization of closed-loop control. The common feature of these systems is that the plant is a continuous-time physical system, and the controller is a discrete-time system (Astrom 1996; Feng and Song 2011; Goodwin et al. 2010). Several engineering applications utilize sampled-data control, for example, robotics (Khosla 1987; Lages and Alves 2006; Faulwasser et al. 2017), haptic devices (Colgate and Schenkel 1997; Salisbury et al. 2004; Sheng and Liu 2004; Culbertson et al. 2018) and power electronics (Liu et al. 2009; Rodriguez et al. 2013; Tahir et al. 2018).

The digital implementation of control strategies introduces time delays to the control loop, which causes stability issues in the resulting system (Khosla 1987; Feng and Song 2011; Zhang and Sun 2013). This problem is further emphasized in distributed and multi-agent systems (Ge and Han 2016; Ge et al. 2017; Liu et al. 2019), where the latency of the communication network is also significant. These systems can be characterized by delay differential equations. Similar engineering problems involving time delay occur in balancing Xu et al. (2016); Balogh et al. (2022), vehicle dynamics (Beregi et al. 2016) and machining processes (Beri and Stepan 2021) as well, resulting in machine tool chatter. In recent years, various new methods have been developed to analyze delayed systems (Insperger et al. 2008; Liu and Kalmár-Nagy 2010).

The design process of sampled-data systems is a well-known task, and different widespread methods are available. One possible approach is to design an entirely continuous-time system that satisfies the specifications of the given task and then discretize the necessary subsystems as described by Kuo (2003). Digital filter design techniques involving pre-warping and bilinear transform are based on this approach

(Nguyen 2016; Xu-bao and Shi-fan 2007), but it is less commonly used in control systems engineering. A different possible approach (see, e.g., Nesić and Teel 2001) is to use a continuous-to-discrete conversion method to determine the discrete-time model of the plant, after which the complete control loop can be considered as a discrete-time system, making it possible to design the controller directly. However, regardless of the applied method, discretization results in information loss about the dynamic behavior between the sampling instants, and the closed control loop can only be modeled in discrete-time.

The lack of continuous-time models of the closed-loop was not considered an issue earlier because digital controllers can operate at high sampling rates. However, Yu et al. (2012) stated that instability problems could occur even with high sampling rates because the appropriateness of a given sampling time cannot be decided only by itself but also concerning the other dynamic characteristics of the controlled system. The relation between sampling time and other parameters can be well expressed using dimensionless units, as shown by Budai et al. (2019) and Budai (2021).

One of the most important problems of sampled-data systems is the intersample ripple which can occur with uniform sampling (Albertos and Salt 2011). This phenomenon can not be modeled with discrete-time models but only in continuous time. However, the main approach to address this problem is not by constructing continuous-time

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models but by avoiding uniform sampling instead. Ge et al. (2017) and Ge et al. (2018) discussed different non-uniform sampling methods, for example random or event-triggered sampling.

The importance of intersample behavior of sampled-data systems further increases with the arrival of Industry 4.0 and cyber-physical systems. In the past few years, various zero-dynamics attack methods have been developed where a distributed control system is compromised by injecting signals that cause harmful vibrations (see, e.g., Teixeira et al. (2012), and Kim et al. (2020)). These attacks are undetectable from the sampled signals; thus, continuous-time models are necessary to detect and eliminate them.

There are well-established methods for converting discrete-time models to continuous-time, and many existing implementations are available, for example, the `d2c()` function in MATLAB and `ToContinuousTimeModel[]` function in Mathematica. However, applying these methods can be problematic: poles of a discrete-time system cannot be mapped uniquely to continuous time, causing restrictions on use (Ogata 1987). Converting systems with negative real-valued poles can be challenging too, requiring further considerations (Kollar et al. 1996). Moreover, Budai et al. (2019) showed that higher-order dynamic behavior could be present in sampled-data mechatronic systems. However, only an approximation method has been presented for obtaining the continuous-time model.

The aim of this paper is to present a method that can be used to model sampled-data control systems in continuous time. The plant and the controller are modeled with continuous- and discrete-time transfer functions respectively, considering uniform sampling and zero-order hold.

This paper is organized as follows. In Section 2, a new approach is presented for matching continuous- and discrete-time system models. An illustrative example is presented in Section 3 to show that the new method is suitable for modeling sampled-data systems in continuous time representing the intersample dynamics, as opposed to classical methods. Finally, Section 4 concludes the paper.

2 Matching continuous- and discrete-time models

This section will present a novel method for converting discrete-time models to continuous-time. Different methods already exist, for example, the zero-order hold (ZOH) method, as illustrated in Figure 1, which is widely used in control system design.

In Figure 1, the continuous-time transfer function $H(s)$ is considered as the equivalent of the discrete-time transfer

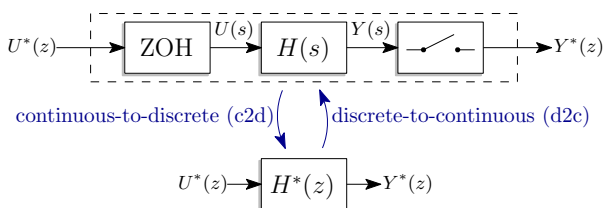


Figure 1. The classical approach for matching continuous- and discrete-time models.

function $H^*(z)$ if the outputs $Y^*(z)$ of the shown systems match for the same input signal $U^*(z)$.

Further on in this paper, we refer to the input signal as U and the output as Y . Discrete-time signals and transfer functions are starred and are defined in z domain, and plain symbols represent their continuous-time counterparts in s domain.

There are different methods to represent discrete-time signals, for example, in the time domain as a sequence or in complex frequency domain using $\mathcal{Z}$ -transform. In case of periodic signals, discrete Fourier transform (DFT) and other related methods can be useful too; see, for example, Press et al. (2007) and Manhertz and Berezsky (2021). These methods can be used to model 2D or higher dimensional data as well, however, in control systems theory usually the time is considered to be the single independent variable.

The main problem stems from that the equivalence is based on discrete-time inputs and outputs, ignoring the intersample behavior of the continuous-time system $H(s)$. This additional information is lost due to sampling, making discrete to continuous (d2c) conversion problematic.

In order to avoid this problem a new approach is presented illustrated by Figure 2 where $H(s)$ and $H^*(z)$ is considered equivalent if the systems in panel (a) and (b) both respond with the same continuous-time output signal $Y(s)$ for any discrete-time input $U^*(z)$.

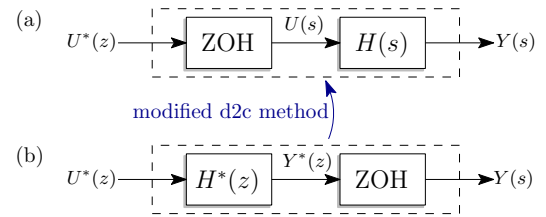


Figure 2. The new approach for converting a discrete-time system model to a continuous-time one. Panel (a) contains a continuous-time transfer function, while panel (b) contains a discrete-time one.

As a result of zero-order hold, this method only handles piecewise constant outputs; however, it can associate exactly one continuous-time system to each discrete-time transfer function. In order to derive the exact conversion formula, we need to examine the effects of zero-order hold.

2.1 Effects of zero-order hold

Consider a discrete-time signal $x[k]$ with $k = 0, 1, 2, \dots$. The continuous-time signal $x(t)$ is produced by a zero-order hold which can be described in time domain as

$$x(t) = x(k\tau + \vartheta) = x[k] \quad \vartheta \in [0, \tau) \quad (1)$$

where τ is the equidistant time step or sampling time, and any $t \geq 0$ can be expressed by a suitable k and ϑ . This is a piecewise continuous function whose Laplace transform is defined as

$$\mathcal{L}\{x(t)\} = X(s) = \int_0^\infty x(t)e^{-st}dt. \quad (2)$$

This improper integral can be calculated separately for each sampling interval as

$$X(s) = \sum_{k=0}^{\infty} \int_0^{\tau} \underbrace{x(k\tau + \vartheta)}_{x[k]} e^{-s(k\tau + \vartheta)} d\vartheta \quad (3)$$

since the integral is well-defined on each subinterval. For further details on the class of Laplace transformable functions, refer to [Schiff \(1999\)](#). By solving the integral above, it yields

$$X(s) = \underbrace{\frac{1 - e^{-s\tau}}{s}}_{H_{\text{ZOH}}(s)} \sum_{k=0}^{\infty} x[k] e^{-ks\tau}, \quad (4)$$

where $H_{\text{ZOH}}(s)$ is often referred by literature as the transfer function of zero-order hold (see, for example, [Ogata \(1987\)](#)). Furthermore, the sum term appears to be the $\mathcal{Z}$ -transform of the original discrete-time signal with the substitution $z = e^{s\tau}$. This substitution is called the *starred transform* as described by [Phillips and Nagle \(1990\)](#).

The $\mathcal{Z}$ -transform of the original signal is defined as

$$\mathcal{Z}\{x[k]\} = X^*(z) = \sum_{k=0}^{\infty} x[k] z^{-k}. \quad (5)$$

It should be noted that if $x[k]$ is a periodic signal, then the substitution $z = \exp(2\pi i \frac{n}{N})$ leads to the discrete Fourier transform (DFT) in the following form

$$\text{DFT}\{x[k]\} = X^*[n] = \sum_{k=0}^{N-1} x[k] e^{-2\pi i \frac{n}{N} k}, \quad (6)$$

where N is the number of samples in a period of the signal. The summation above is only applied to the first N samples and can be efficiently calculated with the fast Fourier transform (FFT) algorithm. FFT methods can easily be extended to handle two or more dimensional data, and they are used to characterize spectral properties of periodic signals ([Press et al. 2007](#)). In contrast, in control system design, signals are usually decaying, and the periodic behavior only appears in marginally stable systems. This paper primarily focuses on 1D data where time is the only independent variable, however; the presented method is not limited to periodic signals and can model arbitrary data.

By using starred transform, Equations (4) and (5) can be combined and simplified to

$$X(s) = \frac{1 - e^{-s\tau}}{s} X^*(e^{s\tau}), \quad (7)$$

which describes the effects of zero-order hold on any signal in s domain.

2.2 Equivalent continuous-time model

In order to match the two systems shown by Figure 2, the output signals have to be equal. The output of system (a) can be expressed as $Y(s) = H(s)U(s)$, where $U(s)$ can be expanded using Equation (7) as follows:

$$Y(s) = \frac{1 - e^{-s\tau}}{s} H(s) U^*(e^{s\tau}). \quad (8)$$

The output signal of system shown by Fig. 2(b) is produced by zero-order hold of $Y^*(z)$, and thus it can be expressed as

$$Y(s) = \frac{1 - e^{-s\tau}}{s} Y^*(e^{s\tau}), \quad (9)$$

which can be further expanded considering the transfer function $H^*(z)$ as

$$Y(s) = \frac{1 - e^{-s\tau}}{s} H^*(e^{s\tau}) U^*(e^{s\tau}). \quad (10)$$

Combining Equations (8) and (10) gives

$$H(s) = H^*(e^{s\tau}). \quad (11)$$

The main conclusion is that the presented approach for converting discrete-time systems to continuous-time leads to the starred transform of the transfer function. Usually, literature introduces starred transform as a method for obtaining the Laplace transform of sampled signals ([Phillips and Nagle 1990](#)), but as Equation (11) shows, this transform can be applied to the system itself. In addition, $H_{\text{ZOH}}(s)$ is canceled out during the simplification, and thus the presented method can be used with higher-order holds as well (More details and examples are provided in Appendix A).

Another important consequence is that if the original discrete-time system $H^*(z)$ is a rational function of z , then $H^*(e^{s\tau})$ will not be a rational function of s . The mapping

$$z = e^{s\tau} \quad (12)$$

between the s plane and z plane makes the continuous-time transfer function $H(s)$ periodic along the imaginary axis, making it difficult to handle.

In order to analyze the dynamic properties of the resulting transfer function $H(s)$, examine how the transform described by Equation (12) maps the poles of the system. Consider $p^* \in \mathbb{C}$, which is a pole of the system $H^*(z)$ on the z plane, in the following form

$$p^* = \varrho e^{j\vartheta}, \quad (13)$$

where $\varrho = |p^*|$ and $\vartheta = \arg(p^*)$ give the polar representation of p^* . The corresponding pole $p \in \mathbb{C}$ of the system $H(s)$ on the s plane can be calculated by using Equation (12) as

$$\varrho e^{j\vartheta} = e^{p\tau}, \quad (14)$$

where the real and imaginary parts of p can be separated as

$$e^{\ln \varrho} e^{j\vartheta} = e^{\text{Re}\{p\}\tau} e^{j\text{Im}\{p\}\tau} \quad (15)$$

resulting in a product on both sides, where the first factor is a real number and the second factor is a complex rotation. The real part of p can be expressed as

$$\text{Re}\{p\} = \frac{\ln \varrho}{\tau} \quad (16)$$

and as for the imaginary part, the solution for the complex rotation results in an infinite number of solutions

$$\text{Im}\{p\} = \frac{\arg(p^*) + 2l\pi}{\tau} \quad \text{with } l \in \mathbb{Z}, \quad (17)$$

where l is an arbitrary integer. This means that $H(s)$ has an infinite number of poles on the s plane corresponding to any given pole of $H^*(z)$ on the z plane. Combining the real and imaginary parts, the final formula for the pole mapping is described as

$$p_l = \frac{\ln |p^*|}{\tau} + \frac{\arg(p^*) + 2l\pi}{\tau}i \quad \text{with } l \in \mathbb{Z}. \quad (18)$$

In literature (Ogata 1987; Wereley and Hall 1991), the s plane is divided to a *primary strip* containing p_0 and infinitely many *complementary strips*, containing all of the other poles as shown by Figure 3.

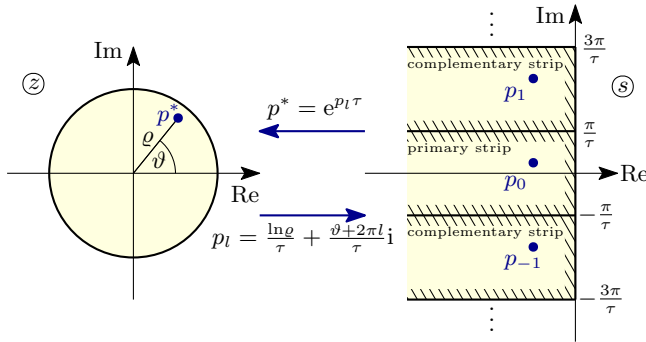


Figure 3. The mapping of poles between the z plane and the s plane.

Conventional methods for converting discrete-time models to continuous time often avoid the poles on the complementary strips for convenience. On the other hand, Budai et al. (2019) showed that the poles on the complementary strip might be responsible for significant vibration components and Budai (2021) specified a condition for deciding whether higher-order dynamic behavior is negligible or not.

2.3 Approximate transfer function

In control systems design, rational functions are the preferred form for transfer functions. However, when the initial system $H^*(z)$ is a rational function of z , then the previously presented method described by Equation (11) results in a system $H(s)$ which is a rational function of $e^{s\tau}$ instead of s . These exponential terms in the denominator of the transfer function result in internal delay loops, which are difficult to handle.

Press et al. (2007) summarized several methods for modeling of data, where the goal is to adjust the model parameters to achieve an optimal approximation of the examined system. A model can be a specific class of function, for example, a rational function in our case. Two possible methods can be considered, Padé and rational Chebyshev approximations.

Padé approximation is widely used to approximate systems where the input is subjected to time delay, as shown by Xing et al. (2016), for example. In contrast, transfer functions in the form of Equation (11) contain delays in the feedback lines as well.

The previously mentioned rational Chebyshev approximations could be another possible approach; however, it can only consider real-valued function of a real variable, as described by Press et al. (2007).

In order to obtain a model which is suitable for control systems design, a different approximation is provided below. Let $\tilde{H}(s)$ denote the approximate transfer function with the following conditions:

1. $\tilde{H}(s)$ is a rational function of s .
2. $\tilde{H}(s)$ is a proper transfer function.
3. $\tilde{H}(s)$ contains an arbitrary number of poles of the original system $H(s)$ with the same corresponding residues to preserve its dynamic properties.
4. $\tilde{H}(s)$ has the same static gain as the original system to preserve its static properties.

In order to satisfy the first two conditions, $\tilde{H}(s)$ is considered in the following form

$$\tilde{H}(s) = \frac{b_n s^n + \dots + b_1 s + b_0}{a_n s^n + \dots + a_1 s + a_0}. \quad (19)$$

This form, however, does not satisfy conditions 3 and 4 directly. For this reason, the following form is presented

$$\tilde{H}(s) = g + \sum_{l=1}^{l_n} \left(\frac{r_l}{s - p_l} + \frac{r_l}{p_l} \right), \quad (20)$$

where $g = \tilde{H}(0)$ is the overall static gain of the system, p_l is a pole of the original system based on Equation (18) and r_l is the corresponding residue. In this form, all the relevant dynamic and static properties directly and separately appear, making it possible to meet the four specified conditions.

In case of $\tilde{H}(s)$ is connected in series with another continuous-time system, Equation (20) could contain higher-order terms too, but only if the overall system remains proper, as later described in Section 3.

The formulae for calculating the g , p_l and r_l parameters are presented below. The static gain g can be easily calculated based on the final value theorem for discrete-time systems as

$$g = H^*(z)|_{z=1}. \quad (21)$$

Each pole p_l of the system can be calculated based on Equation (18) and any number of poles can be taken into account in the approximating transfer function in Equation (20).

Calculating the residues r_l is a nontrivial task because the literature has not paid attention to the mapping of the residues before, but only to the mapping of the poles. In the following, we present the method if the initial discrete-time system is represented by the first order transfer function

$$H^*(z) = \frac{A}{z^{-1}p^* - 1}, \quad (22)$$

where p^* is the pole of the system on the z plane and A is a real-valued gain (note that it is different from the steady-state or static gain). Based on Equation (11), this first order system be converted to continuous time as

$$H(s) = \frac{A}{e^{-s\tau}p^* - 1}. \quad (23)$$

Considering an arbitrary number of poles p_l on the s plane, the corresponding residues r_l can be calculated using the

Heaviside cover-up method as

$$r_l = \text{Res}(H(s), p_l) = \lim_{s \rightarrow p_l} (s - p_l) \frac{A}{e^{-s\tau} p^* - 1}, \quad (24)$$

which is an ∞/∞ type limit because $e^{p_l\tau} = p^*$; thus, by using l'Hôpital's rule, the residues can be expressed as

$$r_l = \lim_{s \rightarrow p_l} \frac{A}{-\tau e^{-s\tau} p^*} = \frac{A}{-\tau e^{-p_l\tau} p^*} = -\frac{A}{\tau}. \quad (25)$$

Note that in this simple first-order case, the values of the residues do not depend on the poles but only on the static gain and the sampling time.

This method can also be used for higher-order systems by decomposing them into first order systems and handling the components separately. Another important remark is that no assumptions have been made regarding p^* ; thus, complex-valued and negative real poles can also be handled.

2.4 Case studies

In order to have a deeper insight into the methods introduced in the previous sections, two examples are provided below.

2.4.1 Example 1: first order system

Consider a discrete-time system described by the transfer function

$$H^*(z) = \frac{-2}{0.5z^{-1} - 1}, \quad (26)$$

where the sampling time is $\tau = 0.1$ s. The parameters of this system can be clearly identified as $A = -2$ and $p^* = 0.5$. Based on Equation (11), the system can be converted to continuous time as

$$H(s) = H^*(e^{s\tau}) = \frac{-2}{0.5e^{-0.1s} - 1} \quad (27)$$

which can be approximated with a rational function $\tilde{H}(s)$. For the approximation, the static gain can be calculated using Equation (21) as

$$g = H(0) = H^*(1) = 4. \quad (28)$$

The dynamic properties of the system are characterized by the poles. The polar coordinates of p^* on the z plane are

$$\varrho = 0.5 \quad \text{and} \quad \vartheta = 0. \quad (29)$$

Equation (12) maps this pole to the s plane and in this example, only the poles corresponding to $l = -2, -1, 0, 1, 2$ are taken into account in the approximate transfer function $\tilde{H}(s)$. The numerical values are shown in Table 1.

Table 1. The poles of the continuous-time system $\tilde{H}(s)$.

l	p_l
2	$-6.93147 + 125.664i$
1	$-6.93147 + 62.8319i$
0	-6.93147
-1	$-6.93147 - 62.8319i$
-2	$-6.93147 - 125.664i$

The residues of the poles of $H(s)$ can be calculated by using Equation (25) as

$$r_l = -\frac{A}{\tau} = 20 \quad (30)$$

and Equation (20) gives the final result

$$\begin{aligned} \tilde{H}(s) = & 1.0277 + \frac{20}{6.9314 + s} \\ & + \frac{277.259 + 40s}{3995.89 + 13.8629s + s^2} \\ & + \frac{277.259 + 40s}{15839.4 + 13.8629s + s^2}, \end{aligned} \quad (31)$$

where the terms corresponding to the complex-conjugate pole pairs are combined to irreducible real-valued partial fractions for convenience. This form can be easily used in various control-system design software tools.

The step response of this approximated function $\tilde{H}(s)$ and the ideal response of $H(s)$ are shown in Figure 4. It is apparent that the approximate system accurately follows the ideal static and dynamic behavior, being only limited by the number of considered poles. Similar to Fourier-analysis, higher-order models with more poles give more accurate results.

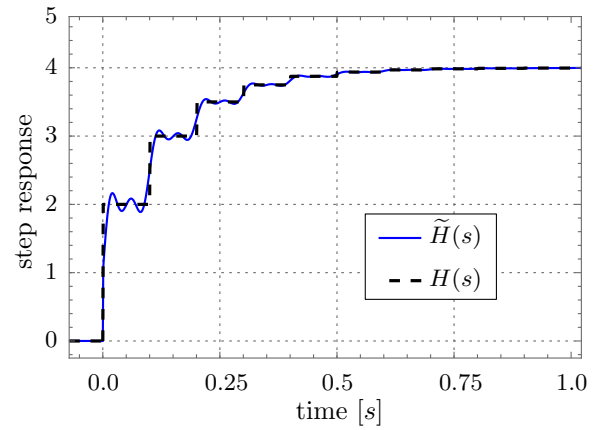


Figure 4. The step response of the first-order discrete-time system $H^*(z)$ with zero-order hold and the continuous-time system $\tilde{H}(s)$.

The transfer functions $H(s)$ and $\tilde{H}(s)$ can be compared in s domain too. Figure 5 shows the absolute values of the transfer functions evaluated at the complex s plane. It is clearly visible that not only the location of the poles are the same, but also their neighborhoods match. The latter is the result of the residue calculation.

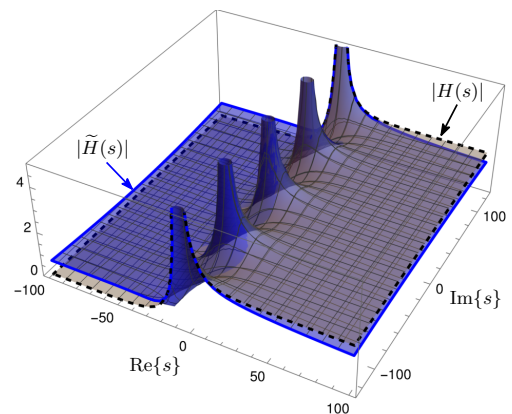


Figure 5. The ideal transfer function $H(s)$ compared with the approximation $\tilde{H}(s)$ in the complex-valued s domain.

Evaluating $H(s)$ and $\tilde{H}(s)$ only at a real-valued subdomain of s gives a section view of the transfer functions at the pole p_0 , illustrated by Figure 6. This view confirms too that the poles and their neighborhoods are calculated correctly, but it is also apparent that at $s = 0$, the static gains are equal.

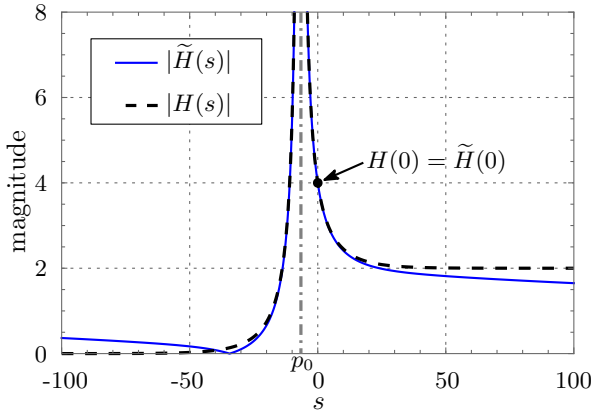


Figure 6. The ideal transfer function $H(s)$ compared with the approximation $\tilde{H}(s)$ in the section $\text{Im}\{s\} = 0$ of the s domain.

2.4.2 Example 2: Negative real-valued poles

Consider a discrete-time system with $\tau = 0.1$ s sampling time, described by the transfer function

$$H^*(z) = \frac{-3}{-0.5z^{-1} - 1}, \quad (32)$$

which has a gain parameter of $A = -3$ and a negative real-valued pole of $p^* = -0.5$ with the polar representation $\varrho = 0.5$ and $\vartheta = \pi$. This system can be converted to continuous-time according to Equation (11) as

$$H(s) = \frac{-3}{-0.5e^{-s\tau} - 1}. \quad (33)$$

The static properties can be described by the steady-state gain $g = H(0) = 2$ and the dynamic properties by the residues of the poles $r_l = -\frac{A}{\tau} = -30$ based on Equation (25). The poles can be mapped to s plane using Equation (18) with $l = -2, -1, 0, 1$ as shown in Table 2.

Table 2. The poles of the continuous-time system $\tilde{H}(s)$.

l	p_l
1	$-6.93147 + 94.2478i$
0	$-6.93147 + 31.4159i$
-1	$-6.93147 - 31.4159i$
-2	$-6.93147 - 94.2478i$

Based on the considered poles, Equation (20) gives the following transfer function

$$\tilde{H}(s) = 1.5516 + \frac{415.888 + 60s}{1035.01 + 13.8629s + s^2} + \frac{415.888 + 60s}{8930.69 + 13.8629s + s^2}, \quad (34)$$

for which the step response is shown in Figure 7. The actual output is shown by the dashed line and the approximation with blue line. The approximate step has the same characteristics as the exact one, and it is possible to further refine the results by higher-order models with higher number of complex-conjugate pole pairs.

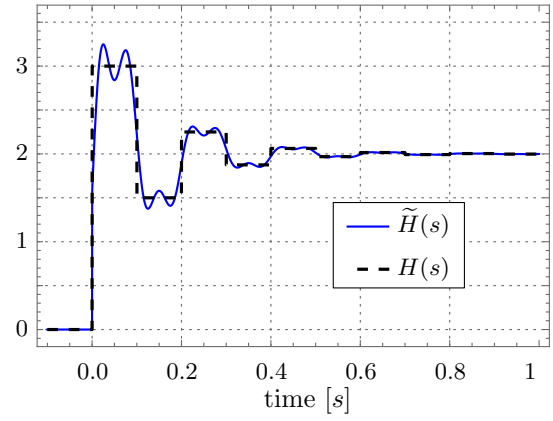


Figure 7. Step response of the discrete-time system with negative real valued pole and zero-order hold.

3 Sampled-data systems

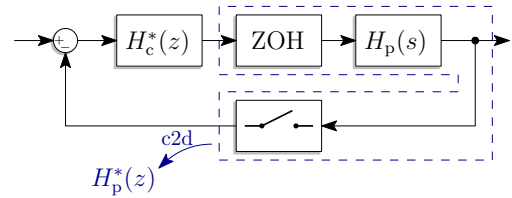


Figure 8. The classical arrangement of a sampled-data control loop.

In this section, the previously presented method is applied to a sampled-data control loop and a numerical example is provided below. Furthermore, another, engineering application is presented in detail in Appendix B. Consider a continuous-time plant which can be described by the transfer function

$$H_p(s) = \frac{1}{s^2}, \quad (35)$$

which, for example, can represent the position of a single degree-of-freedom system with unit mass excited by an input force. Choosing $\tau = 1$ s for sampling time, this system can be discretized with traditional c2d (ZOH or step invariant) methods, which results in the following transfer function

$$H_p^*(z) = \frac{z + 1}{2(z - 1)^2} = \frac{z^{-1} + z^{-2}}{2(1 - z^{-1})^2}. \quad (36)$$

Consider a discrete-time controller represented by the transfer function

$$H_c^*(z) = \frac{3.5 - 3.5z^{-1}}{1 + 0.935z^{-1}}, \quad (37)$$

which is used in a closed control loop shown by Figure 8. The discrete-time model of the closed loop can be calculated as

$$H_{cl}^*(z) = \frac{H_c^*(z)H_p^*(z)}{1 + H_c^*(z)H_p^*(z)} = \frac{1.75 + 1.75z}{0.815 + 1.685z + z^2}. \quad (38)$$

While the discrete-time model only carries information at the sampling points, the exact continuous-time output can be obtained via piecewise continuous numerical simulations. The latter can be considered as the actual output of the

system and it is shown as a dashed black curve by Figure 9. It is apparent that the output converges to a constant value of 1, but the transient behavior shows vibrations with multiple frequencies in the exact piecewise numerical simulation, as similarly shown by Budai et al. (2019).

In digital control systems engineering, it is important to verify whether the control loop satisfies Shannon's sampling theorem. The angular frequency of the sampling is $\omega_s = 2\pi/\tau = 2\pi \text{ rad/s}$ and the characteristic roots of the closed loop are

$$p_{1,2}^* = -0.8425 \pm 0.3243i. \quad (39)$$

The damped natural angular frequency of the system (Ogata 1987) can be calculated as

$$\omega_d = \frac{\omega_s}{2\pi} \arg(p_1^*) = 2.774 \frac{\text{rad}}{\text{s}} \quad (40)$$

and it is clear that the condition for the sampling theorem is satisfied as the inequality

$$6.283 \frac{\text{rad}}{\text{s}} = \omega_s > 2\omega_d = 5.548 \frac{\text{rad}}{\text{s}} \quad (41)$$

is true. The closed loop can be converted to continuous time with a classical discrete-to-continuous (d2c) method, for example the `ToContinuousTimeModel[]` function of Wolfram Mathematica (with "ZeroOrderHold" option) as

$$H_{cl}^*(z) \xrightarrow{\text{d2c}} \frac{7.706 - 0.6889s}{s^2 + 0.2046s + 7.706} \quad (42)$$

whose step response in time domain can be given by using the inverse Laplace transform as

$$y(t) = 1 - e^{-0.1023t} (0.2852 \sin(2.774t) + \cos(2.774t)) \quad (43)$$

which is illustrated as the red curve in Figure 9. The classical method leads to a model where the multi-frequency dynamic behavior is missing, and the step response only gives correct results at the sampling instants (presented by black dots in Figure 9).

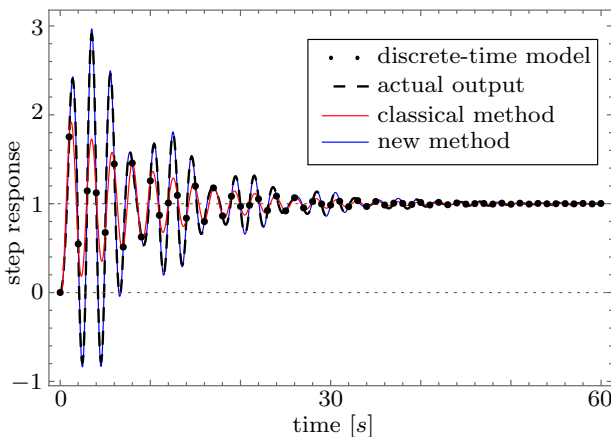


Figure 9. The step response of the different models.

In order to apply the new discrete-to-continuous method, the control loop must be rearranged as illustrated by Figure 10 to isolate a subsystem whose output is piecewise constant. This subsystem converts the reference to the plant's input signal. The corresponding discrete-time transfer

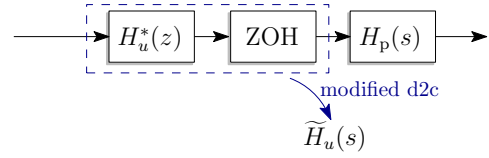


Figure 10. The rearranged sampled-data control loop.

function can be calculated as

$$H_u^*(z) = \frac{H_c^*(z)}{1 + H_c^*(z)H_p^*(z)} = \frac{3.5 - 7z + 3.5z^2}{0.815 + 1.685z + z^2} \quad (44)$$

and it can be converted to continuous time with starred transform as

$$H_u(s) = \frac{3.5 - 7e^{s\tau} + 3.5e^{2s\tau}}{0.815 + 1.685e^{s\tau} + e^{2s\tau}}. \quad (45)$$

In order to determine a rational approximation, the original system must be decomposed to first-order subsystems and a constant gain as

$$H_u^*(z) = 4.294 + \frac{0.3972 + 20.91i}{(-0.8425 - 0.3243i)z^{-1} - 1} + \frac{0.3972 - 20.91i}{(-0.8425 + 0.3243i)z^{-1} - 1} \quad (46)$$

and this way, the individual terms can be handled separately. The poles of the approximate transfer function on the s plane are shown in Table 3. The approximate transfer function is

$$\tilde{H}_u(s) = 3.165 + \frac{-0.7945s + 146.7}{s^2 + 0.2046s + 12.32} + \frac{-0.7945s - 116.1}{s^2 + 0.2046s + 7.706} \quad (47)$$

and in order to determine the final output, the transfer function of the plant is connected in series as

$$H_{cl}(s) = \underbrace{\frac{1}{s^2}}_{H_p(s)} H_u(s). \quad (48)$$

Table 3. The poles of the continuous-time system $\tilde{H}_u(s)$.

p^*	l	p_l
$-0.8425 - 0.3243i$	1	$-0.1023 + 3.509i$
$-0.8425 - 0.3243i$	0	$-0.1023 - 2.774i$
$-0.8425 + 0.3243i$	0	$-0.1023 + 2.774i$
$-0.8425 + 0.3243i$	-1	$-0.1023 - 3.509i$

The approximation used to determine the transfer function $\tilde{H}_u(s)$ only ensured matching static properties with $H_u(s)$. The approximate model of the whole control-loop $H_{cl}(s)$ is sought in the following form

$$\tilde{H}_{cl}(s) = \frac{1}{s^2} \left(\tilde{H}_u(s) + C_1 s + C_2 s^2 \right), \quad (49)$$

which is kept proper even with the additional second-order term $C_1 s + C_2 s^2$ (as it was mentioned previously in Section 2.3) whose real-valued coefficients can be determined with conditions regarding the static behavior of

the system. The first condition is that the final value of the output derivative is zero, which ensures that the output converges to a constant value. This can be expressed by the final value theorem as

$$\lim_{s \rightarrow 0} s \cdot \tilde{H}_{cl}(s) = 0.03486 + C_1 = 0 \quad (50)$$

which is calculated by l'Hôpital's rule and yields the result $C_1 = -0.03486$. The second condition is that the closed control loop has unity gain, which can be expressed similarly as

$$\lim_{s \rightarrow 0} \tilde{H}_{cl}(s) = 0.98598 + C_2 = 1 \quad (51)$$

and the results can be calculated by l'Hôpital's rule as well, resulting in $C_2 = 0.01402$. The final approximate transfer function of the closed control loop is

$$\begin{aligned} \tilde{H}_{cl}(s) = & \frac{3.165}{s^2} - \frac{0.03486}{s} + 0.01402 \\ & + \frac{-0.7945s + 146.7}{s^2(s^2 + 0.2046s + 12.32)} \\ & + \frac{-0.7945s - 116.1}{s^2(s^2 + 0.2046s + 7.706)} \end{aligned} \quad (52)$$

and the corresponding step response is shown as a blue curve in Figure 9.

It can be concluded that the output shows the same dynamic and static behavior as the exact numerical simulation, and it also gives correct results between the sampling intervals. The intersample dynamics of the different models are shown in Figure 11. These results clearly show that the presented method is able to capture the intersample dynamics of the system, which can be attributed to the considered higher-order dynamic behavior.

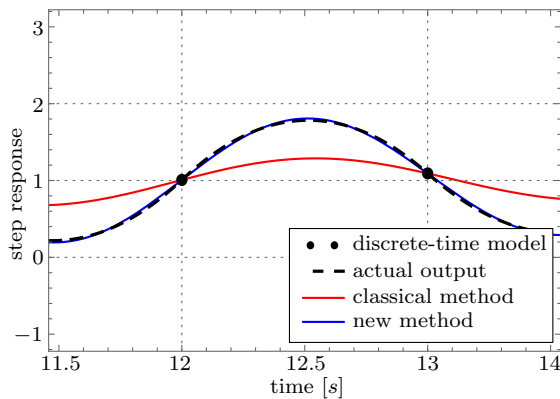


Figure 11. Intersample dynamics of the different models.

4 Conclusions

This paper discussed sampled-data control systems, where the plant is a continuous-time physical system while the controller is in discrete time. The models were considered proper rational transfer functions, and the sampled-data control was applied with uniform sampling and zero-order hold. It was demonstrated that in these systems, multi-frequency vibrations occur that can only be characterized in continuous time. This paper demonstrated that this phenomenon leads to intersample dynamics that classical

models could not handle, even when Shannon's sampling theorem is satisfied. In order to overcome this problem, a new method was presented to provide continuous-time models for the closed control loop. This new approach was derived analytically, and examples for use were also provided. The simulation results showed that this new method is able to accurately model the multi-frequency behavior and the intersample dynamics of the sampled-data control system.

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Appendix A: higher-order holds

The method presented in Section 2 only discussed zero-order hold, but it can also be generalized with higher-order holds.

While zero-order hold fills out intersample time intervals with piecewise constant signals, it is possible to interpolate discrete-time signals with higher-order functions. For example, first-order hold produces piecewise linear output, and an n -th order generalization can be given using cardinal b-splines. According to Unser (2005), the n th order hold can be described by the transfer function

$$H_h(s) = \frac{(1 - e^{-s\tau})^{n+1}}{s^{n+1}\tau^n}. \quad (53)$$

Similarly to Equation (7), the effects of the generalized hold $H_h(s)$ can be described in s domain as

$$X(s) = H_h(s) X^*(e^{s\tau}), \quad (54)$$

where $X^*(z)$ is the $\mathcal{Z}$ -transform of the original discrete-time signal and $X(s)$ is the Laplace transform of the held signal.

Consider a discrete-time system described by the following transfer function

$$H^*(z) = \frac{-6}{0.4z^{-1} - 1}, \quad (55)$$

given in the form of Equation (23) with the parameters $A = -6$ and $p^* = 0.4$. Based on Equation (11), the system can be converted to continuous time as

$$H(s) = H^*(e^{s\tau}) = \frac{-4}{0.6e^{-\tau s} - 1} \quad (56)$$

where $\tau = 0.1$ s is the sampling time. For the rational approximation, the static gain can be calculated using Equation (21) as $g = H(0) = 10$ and the polar coordinates of p^* are $\varrho = 0.4$ and $\vartheta = 0$.

The poles of the original system can be mapped to s plane using Equation (12) according to Table 4.

Table 4. The poles of the continuous-time system $\tilde{H}(s)$.

l	p_l
2	$-9.16291 + 125.664i$
1	$-9.16291 + 62.8319i$
0	-9.16291
-1	$-9.16291 - 62.8319i$
-2	$-9.16291 - 125.664i$

The corresponding residues of $H(s)$ can be calculated by using Equation (25) as $r_l = -\frac{A}{\tau} = 60$, and Equation (20) gives the approximate transfer function

$$\begin{aligned} \tilde{H}(s) = & 3.10988 + \frac{60}{9.1629 + s} \\ & + \frac{1099.55 + 120s}{4031.8 + 18.3258s + s^2} \\ & + \frac{1099.55 + 120s}{15875.3 + 18.3258s + s^2}. \end{aligned} \quad (57)$$

This system can be examined in a configuration similarly as seen in Figure 2, but with different kinds of hold functions which are presented below.

First-order hold

First-order hold (FOH) gives a linear interpolation of a discrete-time signal and it can be described by substituting $n = 1$ into Equation (53) as

$$H_{\text{FOH}}(s) = \frac{(1 - e^{-s\tau})^2}{s^2\tau}. \quad (58)$$

The effects of first-order hold on the original system $H^*(z)$ and its continuous-time approximation $\tilde{H}(s)$ can be examined by comparing their step responses. Figure 12 clearly shows that the outputs are indeed linearly interpolated and the approximate transfer function $\tilde{H}(s)$ matches the actual output.

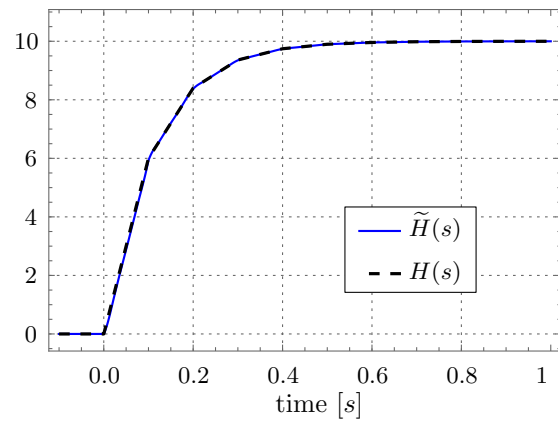


Figure 12. Step response of the first-order system with first-order hold.

Second-order hold

Similarly to the previous case, the second-order hold (SOH) can be described by substituting $n = 2$ into Equation (53) as follows

$$H_{\text{SOH}}(s) = \frac{(1 - e^{-s\tau})^3}{s^3\tau^2}. \quad (59)$$

The step responses are shown by Figure 13, and it is clearly visible that the curves have tangency continuity. It can be concluded that the approximate transfer function $\tilde{H}(s)$ gives satisfactory results in this case too.

Appendix B: application example for modeling zero dynamics attacks

A possible field of application of the presented method is to model zero dynamics attacks, where an attacker can inject signals into the control loop as shown by Figure 14. Such an attack can be carried out by exploiting the zero dynamics (Yuz and Goodwin 2014) of the system, causing unbounded vibrations in the continuous-time plant. However, the attack signal $a[k]$ is chosen so that the perturbations in the sampled output remain arbitrarily small.

An example is provided below based on the results of Kim et al. (2020). Consider a continuous-time plant

$$H_p(s) = \frac{s + 1}{(s^2 + 1)(s^2 + s + 1) + (s + 1)s^2}, \quad (60)$$

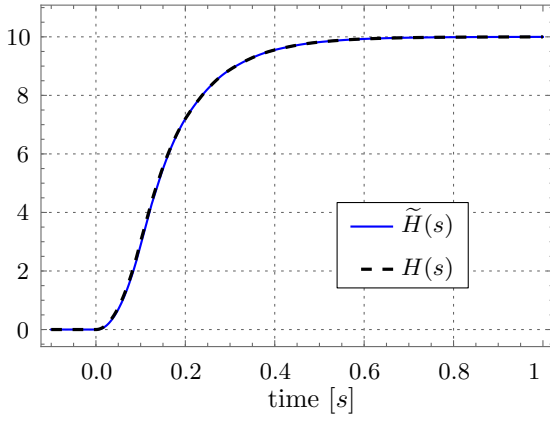


Figure 13. Step response of the first-order system with second-order hold.

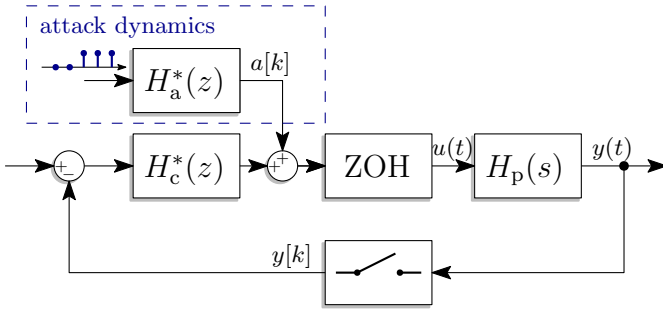


Figure 14. Sampled-data control loop under zero dynamics attacks.

which is sampled with $\tau = 0.1$ s sampling period resulting in the following discrete-time system

$$H_p^*(z) = \frac{0.00016242(z + 3.636)(z - 0.9048)(z + 0.2616)}{(z^2 - 1.987z + 0.9914)(z^2 - 1.804z + 0.8259)} \quad (61)$$

which has an unstable zero at $z = -3.636$, making it susceptible to zero dynamics attacks, and this vulnerability, unfortunately can not be avoided by simply choosing a smaller sampling time (Yuz and Goodwin 2014). In this example, the attack signal $a[k]$ is produced by the step response of a system

$$H_a^*(z) = 10^{-5} \frac{z - 1}{z + 3.636}, \quad (62)$$

which has a matching pole to the unstable zero of the plant resulting in unbounded excitation of the plant. However, because of the pole-zero matching, the feed-forward path can be modeled as

$$H_{ff}^*(z) = H_a^*(z) H_p^*(z) = \frac{1.6242 \cdot 10^{-9} z(z - 0.9048)(z + 0.2616)}{(z^2 - 1.987z + 0.9914)(z^2 - 1.804z + 0.8259)}, \quad (63)$$

where the unstable dynamical behavior cancels out, and only stable poles remain in the system. Note that the attack signal can be scaled down arbitrarily, making the perturbations virtually invisible to the controller. Consequently, the feedback line can be neglected, and the input of the plant is affected by only the attack signal. The attack dynamics can be rewritten as follows

$$H_u^*(z) = H_a^*(z) = -2.75 \cdot 10^{-6} - \frac{1.275 \cdot 10^{-5}}{-3.636z^{-1} - 1} \quad (64)$$

in order to match Equation (23) with the following parameters $p^* = -3.636$ and $A = -1.275 \cdot 10^{-5}$. Equation (18) with $l = -1, 0$ maps the poles to s plane as

$$p_l = 12.9101 \pm 31.4159i \quad (65)$$

and the corresponding residues are given by Equation (25) as $r_l = 1.275 \cdot 10^{-4}$. The steady-state gain based on Equation (21) is $g = 2.75 \cdot 10^{-6}$. The continuous-time effects of the attack can be described by the approximate transfer function based on Equation (20) as

$$\tilde{H}_u(s) = 2.854 \cdot 10^{-6} + \frac{-0.003292 + 0.000255s}{1153.63 - 25.82s + s^2}. \quad (66)$$

Using the new continuous-time model, the feed-forward path can be described as $\tilde{H}_{ff}(s) = \tilde{H}_u(s) H_p(s)$ resulting in the following transfer function

$$\tilde{H}_{ff}(s) = \frac{2.854 \cdot 10^{-6}(63.54s + s^2)}{1153.63 - 25.82s + s^2} \cdot \frac{s + 1}{(s^2 + 1)(s^2 + s + 1) + (s + 1)s^2}. \quad (67)$$

Figure 15 shows the output of the system under zero dynamics attack with different models. The discrete-time model cannot describe intersample dynamics, and the attack remains stealthy, as shown by the black dots. Converting $H_{ff}^*(z)$ to continuous-time with classical d2c methods leads to similar problems as shown by the red line, because the unstable dynamics are canceled out in the original discrete-time model as well. Clearly, for the classical method, the attack always remains hidden.

However, the actual output can be calculated with piecewise continuous simulations (shown by the black dashed line), and the unbounded vibrations can be described with intersample dynamics. The blue line shows the output of the new method, and it is clearly visible that it can accurately model the intersample behavior and detect the zero dynamics attack.

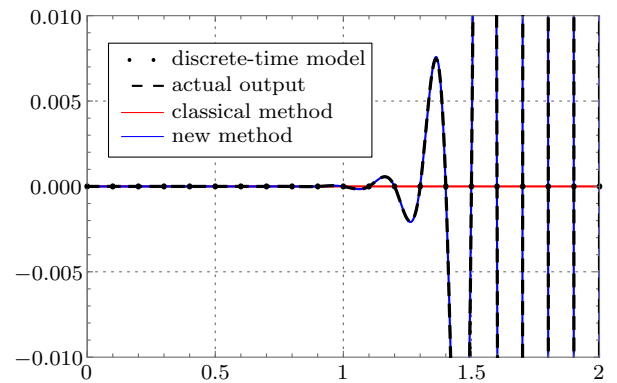


Figure 15. Step response of the system under zero dynamics attack modeled with different methods.