

On the complexity of Stable Hypergraph Matching, Stable Multicommodity Flow and related problems

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Abstract

In this paper we study stable multicommodity flows, stable hypergraph matchings and some related problems, like the Hospital-Resident-Couple (HRC) problem. We give a simpler proof of the existence of a stable multicommodity flow by reducing it to SCARF. We show that finding a stable fractional solution of a HRC or a 3-dimensional stable matching (stable family) instance is PPAD-complete. Then we derive from these results that the FRACTIONAL HYPERGRAPH MATCHING problem is PPAD-complete even if all hyperedge sizes and vertex degrees are at most 3. Furthermore, we prove that computing stable multicommodity flows is PPAD-complete even if the number of commodities is 3, the sum of in and out degrees are at most 3 at each vertex and all terminals and sources are the same. We also show that deciding if there exists a stable integer multicommodity flow is NP-complete even with only two commodities and similar restrictions.

Keywords: PPAD, stable matching, Hospitals/Residents problem, Stable hypergraph matching, Stable flow

1. Introduction

Stable matching problems are widely used in social choice and economics. They represent markets where agents have preferences on each other. Stability means that there are no blocking coalitions, that would mutually like to change
5 their current positions and destabilize the market. Stable matching models are used for example in the National Resident Matching Program (NRMP) in the United States or in the College admission algorithms in Hungary.

There are several generalizations of the stable matching problem. If we allow the vertices (agents) to have capacities larger than one, we get the stable
10 allocation problem. This can be further generalized to include couples on one

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side of the market. This way we obtain the Hospital-Resident-Couple problem. If we have hypergraphs instead of graphs, then we get the stable hypergraph matching problem. Here the vertices rank the hyperedges they are incident with. A special case of this problem is the so-called stable family problem,
15 where the corresponding hypergraph is 3-uniform, 3-partite and each vertex has unit capacity.

Stable flows are another generalization of the stable matching problem, where simple transactions are replaced by series of transactions between different agents, that forward the "flow" between the sender and the receiver.
20 Surprisingly, stable flows can be directly reduced to stable allocation problems, so they can be solved efficiently. A straightforward generalization of the stable flow problem is to introduce multiple commodities, each of which wants to forward its own flow between the agents.

In several cases of these problems it is NP-complete to decide if a stable integer solution exists. However in the previously mentioned problems a fractional
25 solution always exists. Motivated by this, in the recent years there have been many results on so called near-feasible stable solutions (see [1],[2]), where we want to obtain an integral solution that is stable with respect to some modified capacities, that are close to the original ones. For problems like the stable family problem, where each capacity has to be one, this does not make much sense,
30 but for example with the Hospital-Resident-Couple problem it becomes a very relevant question. Nguyen and Vohra proved [2] that in this case we only need to modify the capacities by at most two to always obtain an instance which admits an integral stable matching and also find it algorithmically. Their algorithm
35 however relied on Scarf's algorithm to find a stable fractional solution first, so it did not have polynomial running time. This leads to the question whether a stable fractional solution can be found efficiently. In this paper we show that the answer is no, so even finding a stable fractional solution in the Hospital-Resident-Couple problem is hard, in the sense that it is PPAD-complete.

40 1.1. Related work

The class PPAD was introduced by Papadimitriou [3]. It is a class containing search problems where there always exists a solution and its existence relies on a parity argument in a directed graph. PPAD-complete problems are generally
beleived to be hard.

45 Scarf's Lemma is a fixed point theorem that was proven by Scarf [4], where he also gave an algorithm that finds such a solution, which we will call Scarf's algorithm.

Stable matchings were introduced by Gale and Shapley in 1962 [5]. Since then an enormous amount of research has been done on the field.

50 Aharoni and Fleiner [6] were the first to consider the generalization of stable matchings to hypergraphs. They proved that a stable fractional matching always exists by using Scarf's lemma. Kintali et al. [7] proved that the problem is PPAD-complete even if each hyperedge has size at most 4 and later Ishizuka and Kamiyama [8] proved that it remains PPAD-complete even if the maximum

55 degree in the hypergraph is 3. They also showed that in the case when the maximum degree is two, the problem becomes solvable in polynomial time.

The matching with couples or Hospital-Resident-Couples (HRC) problem has been studied by several authors too, for example Roth [9], McDermid et al. [10], Biró et al [11], [12] and Nguyen and Vohra [2]. Biró et al. proved that
60 the problem of deciding if there exists a stable allocation in a HRC instance is NP-complete, even if the preference lists have length at most two, and each hospital’s capacity is one. Later in [12] they also showed that finding (fractional) stable matchings can be reduced to finding a dominating vector in an instance of Scarf’s lemma.

65 The stable family problem was proposed by Knuth [13] and later investigated by Ng and Hirschberg [14] and Huang [15]. They proved that the problem is NP-complete even with restrictions on the preference lists.

The concept of stable flows is due to Fleiner [16]. He also gave a polynomial-time algorithm to find one, by reducing the problem to a stable allocation in-
70 stance. Later Cseh and Matuschke [17] have made another polynomial-time algorithm that solves the problem directly in a Gale-Shapley like proposal-rejection way. Nguyen et al. [18] showed that even for more general flow concepts a stable flow always exists, by reducing the problem to finding a dominating solution in an instance of Scarf’s lemma, which is known to always exist.

75 Finally, stable multicommodity flows were considered by Király and Pap [19]. They extended the notion of stability from stable flows by introducing ordering on the edges too, that rank the possible commodities that can travel on that edge. These can be interpreted as the transporter’s preferences over what they carry on a given route. In their paper they proved that a stable
80 fractional multicommodity flow always exists, however it is PPAD-hard to find one. Later, it has also been proved by Cseh et al. [17] that deciding if there exists an integer stable multicommodity flow is NP-complete.

1.2. Results

In this paper we further expand the set of PPAD-complete problems with
85 fractional versions and generalizations of stable matching problems. We prove that the FRACTIONAL HYPERGRAPH MATCHING problem is PPAD-complete, even if all edges are of size at most 3, and all degrees are at most 3. These conditions cannot be relaxed further, since if all hyperedges have size at most two, or all degrees are at most two, then the problem becomes polynomial-time
90 solvable. In the first case because the hypergraph becomes a graph and in the second case due to the algorithm of Ishizuka and Kamiyama [8]. However, we prove that deciding if there is an integer solution is still NP-complete if the degrees are at most two.

We prove several results about multicommodity flows too. The way Király
95 and Pap proved the existence of a stable multicommodity flow was by finding a multicolor vertex of a polyhedron. However, this polyhedron had exponentially many facets, so even trying to find a stable flow seemed infeasible in larger networks. In this paper we present another proof of the existence by reducing the

problem to finding a dominating solution in Scarf's lemma. Furthermore the
100 reduction is polynomial and only uses a linear sized matrix (in the number of
commodities, vertices and edges), so it also shows that the problem is in PPAD,
since SCARF is already known to be contained in it. This also means that we can
use Scarf's algorithm on a linear size matrix to find a stable multicommodity
flow, so it should make us able to compute one in a feasible amount of time
105 in most cases. Then we prove that even highly restricted versions of the sta-
ble multicommodity flow problem are PPAD-complete, for example when the
number of commodities is 3 and the sum of in and outdegrees are at most 3.
We also provide a proof of the NP-completeness of deciding whether an inte-
ger solution exists in such an instance even with only two commodities. We
110 show the PPAD-completeness of the fractional versions of some stable matching
problems, such as the HRC and the Stable family problem. As we mentioned
before, these are important observations, since they mean that any algorithm
that tries to find a feasible or near-feasible integer stable matching by rounding
a stable fractional solution cannot be efficient unless PPAD-complete problems
115 are polynomial-time solvable.

1.3. Outline

The rest of the paper is organized as follows: In Section 2 we provide the
basic introduction and definitions to the problems we will consider. In Section
3 we provide a new proof of the existence of a stable multicommodity flow by
120 reducing the problem to SCARF. In Section 4 we prove the PPAD-completeness
of several stable fractional matching problems. Then in Section 5 we show the
NP-completeness of deciding if there exists an integral stable multicommodity
flow even with only two commodities. Finally, in Section 6 we discuss some
open problems.

2. Preliminaries

First, let us state Scarf's Lemma, a fundamental theorem that we will use
throughout our paper.

Lemma 2.1 (Scarf [4]). *Let Q be an $n \times m$ nonnegative matrix, such that every
column of Q has a nonzero element and let $q \in \mathbb{R}_+^n$. Suppose every row i has
130 an strict ordering $>_i$ on those columns j for which $Q_{ij} > 0$. Then there is an
extreme point of $\{Qx \leq q, x \geq 0\}$, that dominates every column in some row,
where we say that $x \geq 0$ dominates column j in row i , if $Q_{ik}x_k = q_i$ and $j \leq_i k$
for all $k \in \{1, \dots, m\}$, such that $Q_{ik}x_k > 0$.*

We define the search problem SCARF the following way:

SCARF

Input: A nonnegative matrix Q and vector q and $>_i$ strict orderings for
135 each row.

Output: An $x \in \mathbb{R}^m$ dominating solution of $\{Qx \leq q, x \geq 0\}$.

The class PPAD

The class PPAD, introduced by Papadimitriou [3] is a complexity class consisting of search problems, where the existence of a solution is guaranteed. Generally, PPAD-hard problems are considered as hard problems, so efficient algorithms for them are unlikely to exist.

When considering polynomial-time reducibility among (search) problems in PPAD, we always mean two polynomial-time computable functions f and g , such that f maps a given instance I_A of problem A to an instance I_B of problem B and if y is a solution for $f(I_A)$ then $g(y, I_A)$ is a solution for I_A .

Stable hypergraph matching

First we define the stable hypergraph matching problem with capacities. Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph. For each $v \in V$ there is a capacity $k_v \in \mathbb{Z}$ and a strict preference list $>_v$ on the hyperedges containing v . Furthermore, for each $e \in \mathcal{E}$, we have a capacity $k_e \in \mathbb{Z}$. Since each hyperedge with capacity larger than one can be substituted with k_e copies of itself, each with capacity one, we can suppose that every k_e is equal to one (the rankings on the edges are extended by ranking the copies among each other arbitrarily).

Definition 2.2. A fractional hypergraph matching is a function $M : \mathcal{E} \rightarrow \mathbb{R}_+$ such that $\sum_{e: v \in e} M(e) \leq k_v$ for all $v \in V$ and $M(e) \leq k_e$ for all $e \in \mathcal{E}$.

Definition 2.3. Given a matching M , a hyperedge f is called blocking, if $M(f) < k_f$ and for each $v \in f$, either v is unsaturated by M or there is an $f_v \in \mathcal{E}$, such that $v \in f_v$, $M(f_v) > 0$ and $f_v <_v f$.

Definition 2.4. A matching M is called stable if there are no blocking hyperedges.

Now we define the search problem of finding a stable fractional matching in a hypergraph.

FRACTIONAL HYPERGRAPH MATCHING

Input: A hypergraph $\mathcal{H} = (V, \mathcal{E})$ with $k_v \in \mathbb{Z}$ capacities and $>_v$ strict orderings for each $v \in V$.

Output: A stable fractional matching M .

It has already been proven by Kintali et al. [7] that there is always a stable fractional matching in any hypergraphic preference system and that finding one is PPAD-complete.

Theorem 2.5 (Kintali et al. [7]). FRACTIONAL HYPERGRAPH MATCHING is PPAD-complete even if all capacities are 1.

The Stable Family problem

The second problem that we will consider in this paper is the 3-dimensional variant of the stable matching problem, also called the stable family problem. It is a special case of the stable hypergraph matching problem with a 3-uniform

3-partite hypergraph and unit edge and vertex capacities. The 3 classes are usually interpreted as men, women and dogs, hence its name. Each man ranks woman-dog pairs, each woman ranks man-dog pairs and each dog ranks man-
175 woman pairs according to some strict preference list which correspond to the rankings of the hyperedges of the vertices. It is known [14] that deciding if there is an integer stable solution in a stable family problem is NP-complete. In this paper we will consider the complexity of the fractional version of this problem. Since it is a special case of the stable hypergraphic matching problem, we know
180 that there always exists a stable fractional solution.

The decision problem we investigate is the following:

F-SFP

Input: An instance I of the Stable family problem.

Output: A stable fractional matching M .

Hospital-Resident-Couple problem

Another closely related problem involving stability is the Hospital-Resident-
185 Couple problem, HRC for short. In this setting there are 3 sets, the hospitals $\mathcal{H}$, the single residents D and the couples C . Each hospital $h \in \mathcal{H}$ has a capacity $k_h \in \mathbb{N}$ and a strict preference $>_h$ on the residents, that is the single residents and the members of couples. Similarly each single resident $d \in D$ has a strict preference $>_d$ on the hospitals, and each couple $c \in C$ has a strict
190 preference list $>_c$ on the ordered pairs of hospitals. These preference lists need not be complete; if a hospital/resident does not appear then we interpret it as being unacceptable. If the lengths of the preference lists are bounded by some constant, then we have the (α, β, γ) -HRC, where each single resident's preference list has length at most α , each couple's preference list has length at most β and
195 each hospital's preference list has length at most γ . The blocking coalitions are defined the following way:

Definition 2.6. *Let M be a matching in the HRC.*

1. *A single resident-hospital pair (d, h) blocks, if d prefers h to its current allocation in M (including the possibility that d is unassigned) and h is
200 either under its quota or has a worse resident in M than d ,*
2. *A coalition (c, h, h') , $h \neq h'$, $c = (c^f, c^m)$ blocks, if c prefers (h, h') to $M(c)$, h is unsaturated or has an at least as bad resident as c^f (since c^f may already be at h , if $h = M(c^f)$) and h' is unsaturated or has an at
least as bad resident as c^m ,*
- 205 3. *A coalition (c, h, h) blocks if $(h, h) >_c M(c)$ and both c^f and c^m are in the best k_h residents for h in the set $M(h) \cup \{c^f\} \cup \{c^m\}$.*

Definition 2.7. *A matching M is stable if it is feasible (so no hospital exceeds its quota and no resident has more than one position) and there are no coalitions that block M .*

210 In this paper we will consider the fractional version of the HRC. This can be seen as the LP relaxation of the problem. In a fractional matching we assign

nonnegative weights $x_{(d,h)}$ and $x_{(c,h,h')}$ to the possible coalitions such that the capacity constraints are still satisfied:

$$\begin{aligned} \sum_{d \in D} x_{(d,h)} + \sum_{c \in C} \sum_{h' \neq h} (x_{(c,h,h')} + x_{(c,h',h)}) + \sum_{c \in C} 2x_{(c,h,h)} &\leq k_h \quad \forall h \in \mathcal{H} \\ \sum_{h \in \mathcal{H}} x_{(d,h)} &\leq 1 \quad \forall d \in D \\ \sum_{h,h' \in \mathcal{H}} x_{(c,h,h')} &\leq 1 \quad \forall c \in C \end{aligned}$$

In this case we have to slightly modify the definition of the blocking coalitions too. To simplify the definition of stability, we extend the rankings of the hospitals on the residents to a ranking on the coalitions containing them. For a hospital h , if two coalitions contain h , then h ranks these two coalitions according to the worst resident in the coalition that is applying to h (this is only relevant when both members of a couple c apply to h). In the case when this resident is the same, h breaks the ties according to the strict preference of the resident/couple that has to be in both coalitions. With this extension in hand, we define stable fractional solutions in the HRC. This definition is motivated by Nguyen et al. [2], because it is the same definition they used in their paper for a stable fractional matching which is then rounded to an integral near-feasible stable matching. They also showed that using this strict preferences for the hospitals, if Scarf's algorithm returns an integer dominating solution, then it is stable.

Definition 2.8. *Let x be a fractional matching in the HRC. A coalition (d, h) , (c, h, h') or (c, h, h) blocks x , if all members of the coalition are either unsaturated by x , or there is a coalition with positive weight in x that they prefer less. x is stable, if there is no blocking coalition to x .*

In a recent paper Biró et al. [12] proved that the HRC problem is reducible to SCARF, so there always exists a stable fractional solution. We introduce the following search problem.

F-HRC

Input: An instance I of the Hospital-Resident-Couple problem.

Output: A stable fractional matching M .

When we use the notation (α, β, γ) -F-HRC we mean that each single resident's preference list has length at most α , each couple's preference list has length at most β and each hospital's preference list has length at most γ . When there are only couples present, we omit α .

Lemma 2.9 (Biró et al. [12]). *The F-HRC problem is polynomially reducible to the SCARF.*

Stable multicommodity flow problem

Finally, let us define the stable multicommodity flow problem. In the stable multicommodity flow problem, we are given a directed graph $D = (V, E)$ with sources $s_1, \dots, s_n$ and sinks $t_1, \dots, t_n$, one for each of the n commodities. Each vertex $v \in V$ has orderings $>_v^j$, $j = 1, \dots, n$ on the edges incident to v . Furthermore, each $e \in E$ has an ordering $>_e$ on the set of commodities. Each edge has $c(e)$ and $c^j(e)$ capacities too, where $c(e)$ is the upper bound on the sum of the flow on e and $c^j(e)$ -s are the upper bounds on the commodity flows $f^j(e)$. Let $\delta^j(v)$ denote the outflow and $\rho^j(v)$ the inflow of a vertex v with respect to f^j . So a flow $f = (f^1, \dots, f^n)$ is called *feasible*, if $\delta^j(v) = \rho^j(v)$ for each $j = 1, \dots, n$ and $v \in V \setminus \{s_j, t_j\}$, $f^j(e) \leq c^j(e)$ for each $e \in E$ and $j = 1, \dots, n$ and $\sum_{j=1}^n f^j(e) \leq c(e)$ for all $e \in E$.

Now we can introduce the notion of stability in multicommodity flows.

Definition 2.10. A walk $W = (v_1, a_1, \dots, v_k)$ blocks with respect to commodity j , if the following four conditions hold:

1. $f^j(a_i) < c^j(a_i)$, $i = 1, \dots, k$,
2. $v_1 = s^j$ or there exists an $u \in V$ such that $f^j(v_1 u) > 0$ and $v_1 u <_{v_1}^j v_1 v_2$, so v_1 likes v_2 more to send commodity j to,
3. $v_k = t^j$ or there exists a $w \in V$, such that $f^j(w v_k) > 0$ and $w v_k <_{v_k}^j v_{k-1} v_k$,
4. if $\sum_{l=1}^n f^l(a_i) = c(a_i)$, then there exists some $j' \neq j$, such that $f^{j'}(a_i) > 0$ and $j' <_{a_i} j$, so if an arc is saturated, then there is some commodity flow on it, that the arc is willing to decrease in order to send more of commodity j .

Definition 2.11. A multicommodity flow is called *stable*, if it is feasible, and there is no blocking walk with respect to any commodity.

The problems we consider are the following.

SMF

Input: An instance I of the stable multicommodity flow problem.

Output: A stable fractional multicommodity flow f .

ISMF

Input: An instance I of the stable multicommodity flow problem.

Question: Is there an integral stable multicommodity flow?

We will also investigate the variants of these two problems where the number of commodities are a fixed constant k . We will denote them as k -SMF and k -ISMF respectively.

3. Reducing smf to scarf

In this section we present a simpler proof to the theorem of Király and Pap [19] about the existence of stable multicommodity flows.

Theorem 3.1. *The SMF problem can be reduced to SCARF in polynomial-time, so in every multicommodity flow instance there always exist a stable fractional solution.*

Proof. The proof will be similar to the proof of Nguyen and Lin in [18] to prove the existence of a generalized stable flow; we only extend it to the multicommodity case. We will construct a matrix (see Figure 1) and apply Scarf's Lemma, then prove that all the dominating solutions correspond to stable multicommodity flows. Then, since a dominating solution always exists, we are done. Let Q be the following matrix: It has $n|E| + 2n|V|$ columns and $(n+1)|E| + 2n|V|$ rows. The columns are indexed by e_i^j , $i = 1, \dots, |E|$, $j = 1, \dots, n$ and these correspond to the $f^j(e_i)$ values in the solution. The remaining $2n|V|$ columns are indexed by $v_i^{j,in}$, $i = 1, \dots, |V|$, $j = 1, \dots, n$ and $v_i^{j,out}$, $i = 1, \dots, |V|$, $j = 1, \dots, n$ and we will need them to maintain that $\delta^j(v) = \rho^j(v)$ for every v, j . The rows are the following:

1. First we add one row for each $e_i \in E$. There are n 1-s in each row, such that $Q_{e_i, e_i^j} = 1$ for $j = 1, \dots, n$, and the other entries are 0. The constraints on the right side are the $c(e_i)$ capacities and the ranking of the columns are the same as the $<_{e_i}$ ranking of the edges.
2. Then we add $n|E|$ rows for each e_i^j that have only one 1-s, such that $Q_{e_i^j, e_i^j} = 1$ and 0 otherwise. Obviously these rows do not need orderings. The constraint on the right side for these rows are the $c^j(e_i)$ capacities.
3. Then we have a row $v_i^{j,in}$ for each $v_i \in V$ and $j = 1, \dots, n$. In the first $n|E|$ columns $Q_{v_i^{j,in}, e^{j'}}$ = 1 if and only if $e = uv_i$ for some $u \in V$ and $j = j'$ and 0 otherwise. Furthermore, $Q_{v_i^{j,in}, v_i^{j,in}} = 1$ and $Q_{v_i^{j,in}, v_i^{j,out}} = 1$ and the other entries are 0. The right side has constraints $q(v_i)$, where $q(v_i)$ is large enough such that at least one of $x_{v_i^{j,in}}$ and $x_{v_i^{j,out}}$ has to be positive in order to achieve equality. (For example summing $c(e)$ on the adjacent edges to v and adding 1 is enough). The ranking on the columns are the following: The worst entry in row $v_i^{j,in}$ is $v_i^{j,out}$, the best is $v_i^{j,in}$ and the others are ranked according to $>_{v_i}^j$.
4. Finally we have a row $v_i^{j,out}$ for each $v_i \in V$ and $j = 1, \dots, n$. Here $Q_{v_i^{j,out}, e^{j'}}$ = 1 if and only if $e = v_i u$ for some $u \in V$ and $j = j'$, 0 otherwise, and $Q_{v_i^{j,out}, v_i^{j,out}} = 1$, $Q_{v_i^{j,out}, v_i^{j,in}} = 1$ same as above. The constraints will be the same $q(v_i)$ -s too. The ranking on the columns are the following: The worst entry in row $v_i^{j,out}$ is $v_i^{j,in}$, the best is $v_i^{j,out}$ and the others are ranked according to $>_{v_i}^j$.

Now we have a polyhedron $\{Qx \leq q, x \geq 0\}$, which satisfies the conditions of Scarf's lemma, so we know that there is some solution x^* that dominates every column. We have to show that $f = (f^1, \dots, f^n)$, $f^j(e) = x_{e^j}^*$ is a feasible flow and that it is stable.

Lemma 3.2. *f is a feasible flow, that is $\delta^j(v) = \rho^j(v)$ for each $j = 1, \dots, n$ and $v \in V \setminus \{s_j, t_j\}$, $f^j(e) \leq c^j(e)$ for each $e \in E$ and $j = 1, \dots, n$ and $\sum_{j=1}^n f^j(e) \leq c(e)$ for all $e \in E$.*

	E^1	E^2	E^n	$V^{1,in}V^{1,out}$	$V^{n,in}V^{n,out}$					
E	I	I		I	0				$c(E)$	
E^1	I	0		0					$c^1(E)$	
E^n	0	0		I					$c^n(E)$	
$V^{1,in}$	A^{in}	0		0	I	I		0	0	$q(V)$
$V^{1,out}$	A^{out}	0		0	I	I		0	0	$q(V)$
$V^{n,in}$	0	0		A^{in}	0	0		I	I	$q(V)$
$V^{n,out}$	0	0		A^{out}	0	0		I	I	$q(V)$

Figure 1: The matrix used in the reduction. A^{in} and A^{out} denote the in and out incidence matrices of D respectively.

320 *Proof.* First of all, because of the first $(n+1)|E|$ rows, the conditions $f^j(e) \leq c^j(e)$ and $\sum_{j=1}^n f^j(e) \leq c(e)$ are trivially satisfied. To show that the inflows and outflows are equal everywhere we need to observe the other rows. Let us take any commodity $j \in \{1, \dots, n\}$ and any vertex $v_i \in V \setminus \{s^j, t^j\}$. We know that since x^* is dominating it dominates both of columns $v_i^{j,in}$ and $v_i^{j,out}$. Because
325 of the construction of the matrix they can only be dominated at row $v_i^{j,in}$ or $v_i^{j,out}$. We have two cases:

- a) If they are dominated at separate rows, then by definition, both row binds (the constraint is satisfied with equality). That means $\delta^j(v_i) = \rho^j(v_i) = q(v_i) - x_{v_i^{j,out}}^* - x_{v_i^{j,in}}^*$, so we are done in this case.
- 330 b) If they are both dominated at, say row $v_i^{j,in}$, then since column $v_i^{j,in}$ is the best for this column, it can only be dominated, if every other x coordinate is zero, where there is a 1 in row $v_i^{j,in}$. So $x_{v_i^{j,in}}^* = q(v_i)$. Then, because of $x^* \geq 0$, and $Q_{v_i^{j,out}} x^* \leq q(v_i)$, and $Q_{v_i^{j,out}, v_i^{j,in}} = 1$ we can see that every other entry is zero in row $v_i^{j,out}$ too. So $\delta^j(v_i) = \rho^j(v_i) = 0$. The case when the columns are
335 dominated in row $v_i^{j,out}$ is analogous. $\square$

Lemma 3.3. f is stable.

Proof. Let us suppose f is not stable and there exists a blocking walk $W = (v_1, a_1, \dots, v_k)$ with respect to some commodity j . Since x^* is a dominating solution, it dominates every column. First, observe that the a_i^j -s cannot be
340 dominated at row a_i^j , since that would mean that they are saturated by f^j , which contradicts the fact that W is a blocking walk. They also cannot be

dominated at the a_i rows, since that would mean that $\sum_{l=1}^n f^l(a_i) = c(a_i)$ and for every $j' \neq j$ for which $f^{j'}(a_i) > 0$, a_i prefers $a_i^{j'}$ to a_i^j , so there is no $j' \neq j$ such that $f^{j'}(a_i) > 0$ and $j' <_{a_i} j$, contradiction.

345 Now let us look at where a_1^j is dominated. If $v_1 \neq s^j$ and it is dominated at row $v_1^{j,out}$, then that means that every entry that $v_1^{j,out}$ prefers less is 0 in x^* . So there is no $v_1 u \in E$ such that $f^j(v_1 u) > 0$ and $v_1 u <_{v_1}^j v_1 v_2$, contradiction. If $v_1 = s^j$, then there is no $v_1^{j,out}$ row, so a_1^j obviously cannot be dominated there.

350 The only possible case left is that a_1^j is dominated at row $v_2^{j,in}$. So by the domination we can see that $x_{v_2^{j,out}}^* = 0$, since it is the worst column for this row, so it is worse than a_1^j too. This also means that $x_{v_2^{j,in}}^* > 0$, since one of them has to be positive, because the corresponding row binds. So if we take a look at a_2^j , it still cannot be dominated at row a_2 or a_2^j , and neither at row $v_2^{j,out}$, 355 since that would mean that every entry that the row prefers less is zero in x^* . But we have just seen that $x_{v_2^{j,in}}^* > 0$ and since it is the worst column it is also worse than a_2^j . So a_2^j can only be dominated at row $v_3^{j,in}$. Continuing using the same argument we can conclude that a_{k-1}^j has to be dominated at row $v_k^{j,in}$. If $v_k = t^j$, then there is no such row, contradiction. If not, then this would mean 360 that there is no $uv_k \in E$, such that $f^j(uv_k) > 0$ and $uv_k <_{v_k}^j v_{k-1} v_k$, because of the domination, so we get a contradiction in this last case too. $\square$

We can conclude that the multicommodity flow given by x^* is a stable multicommodity flow indeed, and as a consequence of Scarf's lemma, one always exists. $\square$

365 4. PPAD-completeness of stable fractional matching problems

In this section we present our hardness results about stable fractional matching problems. First, we state some already known results that we will use.

Lemma 4.1. *SCARF reduces in polynomial time to FRACTIONAL HYPERGRAPH MATCHING.*

370 *Proof.* It is immediate, since both are PPAD-complete. [7] $\square$

Theorem 4.2 (Király and Pap [19]). *The FRACTIONAL HYPERGRAPH MATCHING problem is polynomial-time reducible to the SMF problem with 0-1 capacities.*

We start with a crucial lemma.

375 **Lemma 4.3.** *For every $k \in \mathbb{N}$, we can reduce any instance of k -SMF or k -ISMF in polynomial time to another instance of k -SMF or k -ISMF where every commodity has the same terminal and source, the $>_v^i$ preferences are the same for all i for any $v \in V$, there are no commodity specific capacities and $\rho(v) + \delta(v) \leq 3$ for all non s, t vertices.*

Proof. First we use the reduction used by Cseh [17] to get a version that satisfies the first 3 conditions. Now let v be any non s, t vertex and let $e_1 >_v e_2 >_v \dots >_v e_k$ be its incoming edges and $f_1 >_v f_2 >_v \dots >_v f_l$ be its outgoing edges (after the reduction). Then we substitute each vertex v with a gadget G_v . G_v will consist of vertices $v'_1, \dots, v'_k$ and $v''_1, \dots, v''_l$. There is a $v'_i v'_{i-1}$ and a $v''_i v''_{i+1}$ arc for all i and a $v'_1 v''_1$ arc. These have large enough capacity, such that they cannot be saturated (for example $\sum_{i=1}^k c(e_i) + 1$). These edges can forward any commodity and have arbitrary preferences on them. Then for every $e_i = uv$, $i = 1, \dots, k$ we make an arc $e'_i = u''_j v'_i$ that has the same capacity and preferences on the commodities as e_i , where j is the rank of e_i in u 's preference list. Similarly for each $f_j = vw$ we make an arc $f'_j = v''_j w'_i$, where i is the rank of f_j in w 's ranking. Furthermore, let the rankings of the new vertices be $e'_i >_{v'_i} v'_{i+1} v'_i$ and $f'_j >_{v''_j} v''_j v''_{j+1}$. It is clear that a flow f can be extended to a flow f' in the new graph and conversly any flow f' in the new graph can be reduced to a flow in the original graph by contracting the vertices in the gadgets, since any flow can only enter a gadget on a copy of an original edge and leave on a copy of another original edge. It is also clear that this preserves integrality. Now let us suppose f was stable but there is a blocking walk $W' = (v_1, a_1, \dots, v_k)$ with respect to commodity j to f' . That means that all a_i -s are either unsaturated or there is a worse (for them) commodity flowing through them, $v_1 = s$ or there is an edge $v_1 u$ such that $f'^j(v_1 u) > 0$, $v_1 u <_{v_1} v_1 v_2$ and $v_k = t$ or there is an edge $w v_k$ with $f'^j(w v_k) > 0$ and $w v_k <_{v_k} v_{k-1} v_k$. But this means that the corresponding walk W in the original graph will be a blocking walk too, since if any v'_i copy of v in a gadget has incoming flow from the worse edge than $a_1 = e'_i$, then v has incoming flow from a worse edge than e_i too, since only paths that come from a worse edge can use $v'_{i+1} v'_i$, and analogously for outgoing edges. So W blocks with commodity j , contradiction. The other direction is similar, since if each edge of a walk is unsaturated, or there is a worse commodity on them, then in the new graph this is also true because the $v'_i v'_{i-1}$, $v'_1 v''_1$ and $v''_i v''_{i+1}$ edges cannot be saturated. Also, if there was a flow coming from a worse edge for v than some e , then this flow uses a $v'_i v'_{i-1}$ edge to v_i (for some i , such that e is v 's i -th choice) which is worse for v'_i than e'_i and if there was a flow leaving on a worse edge f for v then this flow leaves v''_j on a $v''_j v''_{j+1}$ edge which is worse for v''_j , than f'_j so we get a blocking walk again, contradiction $\square$

Remark 4.4. By Lemma 4.3 and the fact that the SMF problem is PPAD-hard for unit capacities, we can assume that the $q(v)$ values we need to reduce SMF to SCARF are 4. That means, making $q(v)$ copies of objects in our reduction can be done in polynomial-time. We will still use the notation $q(v)$ instead of 4, to emphasize its meaning.

Lemma 4.5. The SMF is polynomial-time reducible to F-SFP.

Proof. First consider the matrix we used to reduce it to SCARF as shown in Figure 1. First of all we can assume that each commodity has the same source and sink as proven in [17], so for each commodity we have $v^{i,in}$ and $v^{i,out}$ rows

	E^1	E^2	$\dots$	E^n	$V_1^{1,in}$	$V_2^{1,in}$	$\dots$	$V_{q(v)}^{1,in}$	$V_1^{1,out}$	$V_2^{1,out}$	$\dots$	$V_{q(v)}^{1,out}$	$V_1^{2,in}$	$V_2^{2,in}$	$\dots$	$V_{q(v)}^{2,in}$	$V_1^{n,in}$	$V_2^{n,in}$	$\dots$	$V_{q(v)}^{n,in}$	$V_1^{n,out}$	$V_2^{n,out}$	$\dots$	$V_{q(v)}^{n,out}$		
E	I	I			I	0	0			0	0			0	0		0			0	0			0	$\leq$	1
$V^{1,in}$	A^{in}	0			0	I	I			I	I			I	0		0			0	0			0		
$V^{1,out}$	A^{out}	0			0	I	I			I	I			0	0		0			0	0			0		
																									$\leq$	$q(v)$
$V^{n,in}$	0	0			A^{in}	0	0			0	0			0	0		0			I		I	I		I	
$V^{n,out}$	0	0			A^{out}	0	0			0	0			0	0		0			I		I	I		I	
$V_1^{1,in}$	0				I																				$\leq$	1
$V_{q(v)}^{1,in}$																										
$V_{q(v)}^{n,out}$	I				I																				$\leq$	1
E^1																										
E^n																										
$V_1^{1,in}$	I				I																				$\leq$	1
$V_{q(v)}^{n,out}$																										

Figure 2: The matrix obtained after the additions and reorderings of the rows and columns in the proof of lemma 4.5. The top of the matrix can be viewed as an incidence matrix of a 3-partite, 3-uniform hypergraph and each hyperedge capacity is 1.

for the same set of vertices. To reduce the hyperege capacities to 1, instead of one $v^{i,in}$ and one $v^{i,out}$ column, we make $q(v)$ columns of both of them. We will call these $v_j^{i,out}$, $j = 1, \dots, q(v)$. The orderings of the rows will be almost the same, we just have to rank the $q(v)$ columns of $v^{i,out}$ and $v^{i,in}$ among each other, but we can rank these arbitrarily, for example let $v_1^{i,out}$ be the best and $v_{q(v)}^{i,out}$ be the worst. Then we add two new rows for each $v_j^{i,in}$ and $v_j^{i,out}$, $j = 1, \dots, q(v)$ such that there is a 1 in the corresponding column and 0 everywhere else with bounding vector all 1 (so we make two identity matrices). Moreover we add $s^{i,out}$ and $t^{i,in}$ rows (with 1 in the columns corresponding to the appropriate edges and 0 elsewhere), with large enough bounding vectors, such that there cannot be any solution where these rows bind, so no column can be dominated here.

We show that an original dominating solution x can be extended to x' such that it remains a dominating solution and vice versa.

In one direction, we only have to fill up the $v_j^{i,out}$ components starting from the best and always filling up the next in the ranking. Then the e^j columns are obviously still dominated. If a $v_j^{i,in}$ row is fully filled up, then it is dominated at one of the newly added identity matrix rows. If there is a $v_j^{i,in}$ row that is not fully filled up, then $v^{i,in}$ had to be dominated at row $v^{i,out}$ (otherwise $x_{v^{i,in}}$ would be $q(v)$, so every component would be filled up). Since this row still has to bind and the $v_j^{i,in}$ -s are its worst columns, a partially filled up (there can only be at most one) and every empty (by our rule of filling up they have to be worse) columns are dominated here, so all of them are dominated somewhere. The reasoning with the $v_j^{i,out}$ -s is symmetric.

In the other direction we just have to sum up the $v_j^{i,out}$ and $v_j^{i,in}$ components.

If all $v_j^{i,in}$ -s are dominated in the identity matrix, then their sum is $q(v)$, so $v^{i,in}$ is dominated at row $v^{i,in}$. If there is one that is dominated at row $v^{i,in}$ or $v^{i,out}$, then similarly $v^{i,in}$ is dominated here too.

Now we make a couple of observations about this matrix:

1. We can swap the rows of the matrix, such that it has an identity matrix at its bottom.
2. The remaining matrix can be seen as the incidence matrix of a 3-uniform 3-partite hypergraph, because each column has exactly three 1-s. If it is of type e^i , $e = uv$, then the 1-s are at row e , row $v^{i,in}$ and row $u^{i,out}$. If it is of type $v_j^{i,in/out}$, then there is a 1 at $v^{i,in}$ -s row, $v^{i,out}$ -s row, and at one of the newly added $v_j^{i,in/out}$ rows, that is not moved to the bottom identity matrix. Furthermore, we can see that if we partition the rows to $E \cup \{\text{newly added rows}\}$, $v^{i,in}$ rows and $v^{i,out}$ rows, then each column (hyperedge) contains exactly one vertex from each class. It is important to note that here the $s^{i,out}$ and $t^{i,in}$ rows are counted among the $v^{i,in}$ and $v^{i,out}$ rows respectively, not among the newly added rows.
3. Since the stable multicommodity flow problem is still PPAD-hard if all edge capacities are 0 or 1, and each added $v_j^{i,in/out}$ row has upper bound 1, we can assume that the bottom identity matrix has the all 1 bounding vector, so in our hypergraph all hyperedges have unit capacity, therefore we have a FRACTIONAL HYPERGRAPH MATCHING instance with $k_e \equiv 1$.

The new matrix is illustrated in figure 2.

Let the preferences of the vertices on the hyperedges be the same as the corresponding row's preferences on the nonzero columns. What remains to be shown is that the stable matchings of the 3-partite hypergraph correspond to the stable solutions of the SCARF instance.

Let us suppose that we have a stable fractional matching, but the corresponding vector is not a dominating one. Then there has to be a column that is not dominated at any row. But this means that each of the 3 rows that could dominate are either nonbinding, or there are less preferred columns with nonzero value. So the edge corresponding to the column is a blocking edge, contradiction.

However, we are still not completely done, since in a F-SFP instance we need that every vertex has capacity 1 too. We will construct another 3-partite hypergraph from the previous one in the following way: Let the 3 classes be X , U and V with $X = E \cup \{\text{newly added rows}\}$. Take each vertex v of the hypergraph that has capacity $q(v) > 1$ (they can only be from U or V) and make $q(v)$ copies of that vertex, that will all have capacity 1. (We can assume that $q(v)$ is integer). Then for each (x, u, v) hyperedge we make (x, u_i, v_j) hyperedges for all $i = 1, \dots, q(u)$, $j = 1, \dots, q(v)$. Let the u_i vertices' ranking of the hyperedges be the same as the u vertices' ranking with the extension that for a given hyperedge (x, u, v) , $(x, u_i, v_1) >_{u_i} (x, u_i, v_2) \dots$. Similarly for the v_j -s. The x vertices' preferences are extended in such a way that if $(x, u, v) >_x (x, u', v')$, then $(x, u_i, v_j) >_x (x, u'_k, v'_l)$ for all i, j, k, l . Furthermore, we rank

the (u_i, v_j) pairs in an arbitrary way. For example, $(x, u_1, v_1) >_x (x, u_1, v_2) >_x (x, u_2, v_1) >_x (x, u_1, v_3) \dots$

Now let us suppose that we have a stable fractional matching in this new graph. We obtain a matching for the original hypergraph by summing on the hyperedges corresponding to the same original one. Since each of those contain the same x vertex with capacity 1, this new matching will be feasible too. Now let us suppose that there is a blocking hyperedge (x, u, v) . So x is unsaturated or prefers (u, v) to some (u', v') , where it is matched with some positive value. This means that x is unsaturated or there is an (u'_k, v'_l) where x is matched with positive value, that it prefers less than any (u_i, v_j) . Similarly u is unsaturated or there is a v' , such that $(x', u, v') <_u (x, u, v)$, and (x', u, v') has positive value in the matching. In the first case, there has to be an i , such that u_i is unsaturated, and in the second case if $(x, u, v) >_u (x', u, v')$, then $(x, u_i, v_j) >_{u_i} (x', u_i, v'_k)$ for any i, j, k , so there is at least one (i, k) pair that is both worse and has positive value. Using the same argument for v , we get that there is a pair i, j , such that (x, u_i, v_j) is a blocking edge, contradiction.

Since this hypergraph is 3-partite and had all one edge and vertex capacities, it corresponds to a stable family instance, so we reduced the problem to F-SFP. $\square$

Lemma 4.6. *The SMF is polynomially reducible to F-HRC.*

Proof. First we reduce it to the 3-partite stable hypergraph problem with vertex capacities as before (so here we do not make $q(v)$ copies of the vertices in U and V). In other words, we take the stable hypergraph instance corresponding to the matrix from figure 2.

Then we observe that, since we can assume that the aggregate capacities of the stable multicommodity flow were all 0 or 1, and the newly added $v_j^{i,in/out}$ rows have capacity 1 too, all vertices in the $X = E \cup \{\text{newly added rows}\}$ class of our hypergraph have capacity 1. Now let the elements of the X class be the couples, and the hospitals will consists of the union of the other two classes' vertices.

The ranking of the couples on the pairs of hospitals will be the same as the corresponding vertex's ranking on the hyperedges containing it. (The other hospital pairs will be considered unacceptable to the couple). Now observe that any two vertices with one from X uniquely determines a hyperedge, if there is any, because e ($e = uv$) and $u^{i,out}$ uniquely determines the last vertex, that is $v^{i,in}$, and vice versa, so the ranking on the hyperedges containing a vertex v gives a unique ranking on the vertices of X adjacent to v . So the hospital's ranking on the couples will be the following: $c > c'$ for h , if $(c, h, *) >_h (c', h, *)$ in the hypergraph, where $(c, h, *)$ denotes the unique hyperedge containing c and h . Furthermore, let the hospital's capacities be the $q(v)$ capacities of the vertices.

Now let us suppose we have a stable matching in the F-HRC instance. Assume that there is a blocking hyperedge (x, u, v) in the 3-partite stable hypergraph problem. Then x prefers (x, u, v) to some (x, u', v') , where it is matched with

535 some positive value meaning $(u, v) >_x (u', v')$ or it is unsaturated. u is either unsaturated, or there is a hyperedge (x', u, v') , that it prefers less (here v could be v' , but $x \neq x'$ because of our observation). But $(x, u, v) >_u (x', u, v')$ means that in the hospital's ranking, u prefers (the corresponding member of) x more than x' . Similarly either v is unsaturated, or there is an x' at v , that it prefers
540 less than x . So (x, u, v) would form a blocking coalition, contradiction. $\square$

As a consequence, we have the following theorems:

Theorem 4.7. *The F-HRC problem is PPAD-complete, even if each hospital only receives applications from men or women.*

Theorem 4.8. *The F-SFP problem is PPAD-complete.*

545 **Theorem 4.9.** *The SMF problem is PPAD-complete.*

Theorem 4.10. *FRACTIONAL HYPERGRAPH MATCHING is PPAD-complete even if each hyperedge has size at most 3, and each vertex has degree at most 3, each capacity is 1 and the hypergraph is polynomial-time 3 edge-colorable.*

Proof. For this we will use that since F-SFP is PPAD-complete, it is enough to
550 apply the reduction used by Ishizuka and Kamiyama [8] to those hypergraphs that correspond to an SFP instance, because we can reduce any hypergraph to a stable family instance as a first step. For the sake of transparency we describe their construction here too.

Let $\mathcal{H} = (V, \mathcal{E})$ be a stable-family hypergraph. Make a new hypergraph
555 $\mathcal{H}' = (V', \mathcal{E}')$ the following way. The vertex set of $\mathcal{H}'$ will be $V' = \{v_i : v \in V, i = 1, \dots, \deg(v)\} \cup \{v'_i : v \in V, i = 1, \dots, \deg(v) - 1\}$. For each $e \in \mathcal{E}$ we make a hyperedge $e' = \{v_{r(v,e)} : v \in e\}$, where $r(v, e)$ is 1, if e is v 's best, 2 if it is v 's second, $\dots \deg(v)$ if it is v 's worst. So for a hyperedge e we make a hyperedge e' consisting of the $r(v, e)$ -th copies of the v vertices in e . Let $E' = \{e' : e \in \mathcal{E}\}$.
560 Then we define the edge set of $\mathcal{H}'$ as $\mathcal{E}' = E' \cup \{\{v_i, v'_i\}, \{v'_i, v_{i+1}\} : v \in V, i = 1, \dots, \deg(v) - 1\}$. The new rankings are not important for us, so we do not describe them here. What is important is that because each $e \in \mathcal{E}$ had at most 3 vertices, so does E' and so does $\mathcal{E}'$, therefore both the degree and the hyperedge size constraints are satisfied in the new instance. It only remains to prove that
565 it can be edge-colored by 3 colors in polynomial time. For this, we will use the following edge coloring: color each hyperedges in E' red, each hyperedge of the form $\{v_i, v'_i\}$ green, and each $\{v'_i, v_{i+1}\}$ yellow. Since for every vertex v only one hyperedge can be it is i -th in the ranking, we get that each vertex can be only adjacent to at most one red edge. It is easy to observe that each vertex
570 can only be in one $\{v_i, v'_i\}$ edge and one $\{v'_i, v_{i+1}\}$ edge too, so the coloring is good indeed, and it can be done in linear time. $\square$

Now we can prove that even a restricted version of 3-SMF is PPAD-complete:

Theorem 4.11. *3-SMF is PPAD-complete even if all capacities are 0 or 1, all commodities have the same source and terminal and the subgraphs corresponding
575 to the edges that can forward a commodity form disjoint st paths.*

Proof. We will use almost the same reduction here as Király and Pap [19]. Since we have proven that FRACTIONAL HYPERGRAPH MATCHING is PPAD-complete even if the hypergraph is 3 edge-colorable, we only need to reduce hypergraphs with this property. But before we make the reduction, we color the edges of the hypergraph with 3 colors as mentioned before. The trick is that now we will only need 3 commodities, one for each color.

Let us describe the reduction itself. Let our (3-colorable) hypergraph be $(V, \mathcal{E})$, $V = \{v_1, \dots, v_n\}$. Now the new graph for the 3-SMF instance will have the following vertices: for each $v_i \in V$ we have two vertices v'_i and v''_i . Furthermore we make a source s and a sink t . The edges will be the following: For each i we add an edge $e_i = v'_i v''_i$ with capacity 1, that can only forward the commodities corresponding to the colors of the edges adjacent to v_i (so other commodity capacities are 0). The ranking of e_i on the relevant commodities is the same as the ranking of v_i on the adjacent hyperedges. Then for each hyperedge $e = \{v_j, v_k, v_l\}$ (we can suppose the hypergraph is 3 uniform by padding smaller hyperedges with dummy vertices) we add sv'_j , $v''_j v'_k$, $v''_k v'_l$ and $v''_l t$ edges that can only carry the commodity corresponding to the color of e .

Now observe that because of the properties of the coloring, for each commodity the edges that it can travel on form internally disjoint st paths. So each commodity i can only flow on a path corresponding to a hyperedge of color i , and for each hyperedge of color i we can send a flow of commodity i on the corresponding path. So the fractional flows correspond to fractional matchings and vice versa. Furthermore, a blocking hyperedge would mean that the corresponding path is blocking, and a blocking st path would mean that the corresponding hyperedge is blocking. Since a blocking walk can only start from s and end in t in this graph, we are done. $\square$

5. The NP-completeness of 2-ismf

Now we turn our attention to the integral case and prove that deciding if there exists an integral stable multicommodity flow is hard even if we have only 3 commodities. First, we mention that the containment of ISMF in NP is already known:

Lemma 5.1 (Király and Pap [19]). *ISMF is in NP.*

We will use the following theorem of Biró et al. [11]:

Theorem 5.2 (Biró et al. [11]). *(2,2)-HRC is NP-complete, even if the following conditions hold:*

- I) *Each hospital has capacity one,*
- II) *Each hospital appears at most once in every couple's preference list,*
- III) *There are no single residents,*
- IV) *Most hospitals receive applications only from men or women (we will call these man; and woman-Hospitals respectively) and the hospitals for which this is not true can be grouped to triangles (h_1^i, h_2^i, h_3^i) with couples c_1^i, c_2^i, c_3^i such that they only receive application from c_1^i, c_2^i, c_3^i and the preference lists are the*

following:

$$\begin{aligned}
c_1^i : (*, *) &> (h_1^i, h_2^i) \\
c_2^i : (*, *) &> (h_2^i, h_3^i) \\
c_3^i : (*, *) &> (h_3^i, h_1^i) \\
h_1^i : c_3^i &> c_1^i \\
h_2^i : c_1^i &> c_2^i \\
h_3^i : c_2^i &> c_3^i,
\end{aligned}$$

where $*$ -s denote man or woman-hospitals. Furthermore each man; or woman-hospital only receives at most one application from these triangles.

Since we will need the details of the construction of their reduction, we describe it here. They reduce the problem (2,2)-E3-SAT, that is the problem of deciding if a given Boolean formula φ in CNF over a set of variables V is satisfiable, where φ has the following properties: (i) each clause contains exactly 3 literals and (ii) for each $v_i \in V$, each of literals v_i and $\bar{v}_i$ appears exactly twice in φ . Let $V = \{v_1, \dots, v_n\}$ be the set of variables and $C = \{c_1, \dots, c_m\}$ be the set of clauses. Then the couples of the HRC instance will be $A \cup B$, where $A = \{(a_i^r, b_i^r) : r = 1, 2, i = 1, \dots, n\}$ and $B = \{(c_j^s, d_j^s) : s = 1, 2, 3, j = 1, \dots, m\}$. The hospitals will be $H \cup T$, where $H = \{h_i^r : r = 1, \dots, 6, i = 1, \dots, n\}$, $T = \{t_j^r : r = 1, \dots, 6, j = 1, \dots, m\}$. Define $h(d_j^s)$ to be h_i^{2r+1} if the r -th appearance of v_i is at position s at c_j , and h_i^{2r+2} if the r -th appearance of $\bar{v}_i$ is at position s at c_j . Similarly let $d(h_i^{2r+1})$ be d_j^s if the r -th occurrence of literal v_i is at the s -th position of c_j and $d(h_i^{2r+2})$ be d_j^s if the r -th occurrence of literal $\bar{v}_i$ is at the s -th position of c_j . So the possible applications, with the order of the preferences are the following:

$$\begin{aligned}
(a_i^1, b_i^1) : (h_i^1, h_i^3) &> (h_i^2, h_i^4) \\
(a_i^2, b_i^2) : (h_i^2, h_i^5) &> (h_i^1, h_i^6) \\
(c_j^1, d_j^1) : (t_j^4, h(d_j^1)) &> (t_j^3, t_j^1) \\
(c_j^2, d_j^2) : (t_j^5, h(d_j^2)) &> (t_j^1, t_j^2) \\
(c_j^3, d_j^3) : (t_j^6, h(d_j^3)) &> (t_j^2, t_j^3).
\end{aligned}$$

Theorem 5.3. 3-ISMF is NP-complete even if all capacities are 0 or 1, all commodities have the same source and terminal and the subgraphs corresponding to the edges that can forward a commodity form disjoint st paths.

Proof. The problem is in NP by Lemma 5.1, so we only have to prove NP-hardness. We will prove that the instances we can get in the above reduction can be further reduced to an instance of a FRACTIONAL HYPERGRAPH MATCHING problem, where the hypergraph is 3 edge-colorable. Then we are done, because we can use the same reduction as in the proof of Theorem 4.11, since it preserves integrality too and its two-way, so stable matchings correspond to stable flows and vice versa. Let our hypergraph be $\mathcal{H} = (V, \mathcal{E})$, where $V = A \cup B \cup H \cup T$, and the hyperedges are the possible couple-hospital triples. Since in the HRC instance, each couple only applies to a hospital in one of their choice, so the rankings of the hospitals on the residents can be extended to a ranking on the couples (since from every couple only one of them applies to any given hospital).

This also means that any hospital and couple vertex uniquely defines a hyperedge in $\mathcal{E}$ if one exists. So the preferences of a hospital h on the couples can be directly extended to the preference of vertex h on the adjacent hyperedges. The couples' preferences are easily extended too. Now it is easy to observe that
665 integral stable matchings of the two instances correspond to each other, since a blocking (c, h, h') coalition would mean a blocking hyperedge and vice versa. So again, it only remains to prove that we can color the hyperedges of $\mathcal{H}$ by 3 colors. First, we color the edges that contain couples of the form (c_j^s, d_j^s) . We claim that we can color these in such a way that any such edge containing
670 a hospital of the form h_i^3, h_i^4, h_i^5 or h_i^6 will be green or red. But this can be seen from the fact that these couples can be partitioned to groups of 3, whose hyperedges only intersect other edges (and only ones containing (a_i^r, b_i^r) couples) in the h_i^3, h_i^4, h_i^5 and h_i^6 vertices. Also in any group a hospital of this form can only be in one hyperedge, since we know it is in at most two, and
675 one of them contains an (a_i^r, b_i^r) pair. So we can just color two of the 3 hyperedges of the form $((c_j^s, d_j^s), t_j^{s+3}, h(d_j^s))$ green and the third red and we can still color the remaining $((c_j^1, d_j^1), t_j^3, t_j^1), ((c_j^2, d_j^2), t_j^1, t_j^2)$ and $((c_j^3, d_j^3), t_j^2, t_j^3)$ edges properly. Now the remaining edges form disjoint quartets, apart from the $h_i^{\geq 3}$ vertices. But we now know that on every h_i^3, h_i^4, h_i^5 and h_i^6 there is only green
680 and red hyperedges. It is straightforward to verify that in every case we can color the $((a_i^1, b_i^1), h_i^1, h_i^3), ((a_i^2, b_i^2), h_i^2, h_i^5), ((a_i^1, b_i^1), h_i^2, h_i^4)$ and $((a_i^2, b_i^2), h_i^1, h_i^6)$ hyperedges properly with 3 colors. For example when h_i^3 and h_i^4 are red and h_i^5 and h_i^6 are green the following coloring will be appropriate: $((a_i^1, b_i^1), h_i^1, h_i^3)$ and $((a_i^2, b_i^2), h_i^2, h_i^5)$ yellow, $((a_i^1, b_i^1), h_i^2, h_i^4)$ green and $((a_i^2, b_i^2), h_i^1, h_i^6)$ red. $\square$

Remark 5.4. By this reduction, observe that we also have proven that deciding
685 if an instance of FRACTIONAL HYPERGRAPH MATCHING has an integral solution is NP-complete, even if the maximum degree is two, each edge has size at most 3 and the edges are 3-colorable.

If the number of commodities are only two, then if we assume that the
690 two sources and terminals are the same, and the edges corresponding to one commodity form disjoint st paths, then the problem always admits a stable integral multicommodity flow, and it can be found in polynomial time.

Lemma 5.5. 2-ISMF and 2-SMF can be solved in polynomial time by Algorithm
695 1 if the two sources and terminals are the same, and the edges that can forward a given commodity form disjoint st paths.

Proof. Since in a network like this every blocking walk has to be an st path, we only have to check the stability regarding these. But if an st path would block with respect to commodity 1, then the path is undersaturated by commodity 1, so there was a step when we decreased a flow, but in the last such step, an
700 edge that prefers commodity 2 became saturated, so the path cannot block. A path also cannot block with commodity 2, because otherwise we would have sent more flow of commodity 2 in the steps corresponding to that path. $\square$

An immediate consequence is the following:

Algorithm 1

Saturate all st paths corresponding to commodity 1 with commodity 1.
Let $P_1, \dots, P_k$ be the st paths that can forward commodity 2.
for $i = 1, \dots, k$ **do**
 while P_i is not saturated with commodity 2 and all saturated edges $e \in P_i$
 that forward a positive value of commodity 1 prefer commodity 2 **do**
 increase the flow of commodity 2 on P_i by 1,
 decrease the flow of commodity 1 on the commodity 1 st paths that
 became oversaturated by 1.
 end while
end for

Theorem 5.6. *Every instance of the FRACTIONAL HYPERGRAPH MATCHING where the hypergraph is 2 edge-colorable always admits an integral stable matching and it can be found in polynomial time.*

Remark 5.7. *The algorithm of Ishizuka and Kamiyama also works for the 2 edge-colorable case, since then all degrees are at most 2, but only if each capacity is 1. Furthermore, their algorithm only returns a half integer solution and it can happen that it returns a fractional solution even if there is an integer one. Their algorithm basically does the following: if a hyperedge is most preferred by everyone, then it is added to the matching and all vertices inside and intersecting hyperedges are deleted. If this procedure terminates, then it adds all remaining hyperedges with 0.5 weight. It is easy to check that for example in an even cycle, where everyone prefers its left neighbor, there exists a stable matching, but the algorithm of Ishizuka and Kamiyama returns the fractional matching with all edge weights set to 0.5.*

If we do not require the edges which can forward a given commodity to form disjoint st paths, then we can further reduce the number of commodities needed for NP-completeness to two.

Theorem 5.8. *2-ISMF is NP-complete even if all capacities are 0 or 1.*

Proof. We will prove this by reducing the version of (2,2)-HRC described above. Let $\mathcal{I}$ be an instance of HRC. We will call a couple c *ordinary* if c^m only applies to man-hospitals and c^w only applies to woman-hospitals. We will construct an instance $\mathcal{I}'$ of 2-ISMF the following way: First of all let us make a source vertex s that will be the source of every commodity, and a sink vertex t that will be the sink of every commodity. Then, for each couple c we assign vertices $v(c^m)$ and $v(c^w)$ and an edge $e(c) = v(c^m)v(c^w)$ with $c(e(c)) = c^1(e(c)) = c^2(e(c)) = 1$. If c is ordinary, then let $e(c)$ s ranking on the commodities be $1 >_{e(c)} 2$ (for non-ordinary couples we will clarify this later). Next for every man-hospital h , we assign a vertex $v(h)$ and an edge $e(h) = sv(h)$ with $c(e(h)) = c^1(e(h)) = c^2(e(h)) = 1$. Then, since h can appear at most twice on the preference lists of couples, we make one or two edges from $v(h)$ to the $v(c^m)$ vertices of these

couples. If c is an ordinary couple, then the edge $v(h)v(c^m)$ can only carry commodity 1 (so other capacities are 0) if h is in c 's first choice, and commodity 2 if its in c 's second choice. If c is not ordinary, then the edge $v(h)v(c^m)$ can forward only one commodity too, that will be clarified later in the description of the gadgets. Now if the two edges leaving $v(h)$ can carry only different commodities, then there is no need to rank the outgoing edges, but we rank the commodities on $e(h)$ in such a way that the commodity that can travel to the better couple for h is first and the other is second. If the two edges can only carry the same commodity, then we do not need rankings on $e(h)$, since only one commodity will be able to travel on it, but we rank the two outgoing edges the same way as h ranks the two couples they are pointing to. Similarly, we make vertices $v(h)$, and edges $v(c^w)v(h)$ and $v(h)t$ for all woman-hospitals. The rankings, incoming edges and capacities are made symmetrically.

For a triumvirate (c_1, c_2, c_3) consisting of non-ordinary couples we make the following gadgets (see Figure 3):

We make four vertices u_{12}, v_{12}, u_3, v_3 and edges $su_{12}, u_{12}v(c_1^m), u_{12}v(c_2^m), v(c_1^w)u_3, v(c_2^w)v_3$ and $v_{12}t$ each with capacities $c(e) = c^1(e) = 1, c^2(e) = 0$. We add further edges $sv(c_3^m), v(c_3^w)u_3, v_3t$ with capacities $c(e) = c^2(e) = 1, c^1(e) = 0$. Finally we add an u_3v_{12} edge with $c(u_3v_{12}) = c^1(u_3v_{12}) = c^2(u_3v_{12}) = 1$, with ranking $2 > 1$ and a $v_{12}v_3$ edge, which can forward commodity 1 from v_3 to v_{12} and commodity 2 from v_{12} to v_3 (but only one unit of flow can go through it), with ranking $1 > 2$. We can implement this two-way edge with a subgadget G' the following way: make two more vertices v'_{12} and v'_3 and add directed edges $v_3v'_3$ and $v'_{12}v_{12}$ with unit capacity that can only carry commodity 1 $v_{12}v'_3$ and $v'_{12}v_3$ with unit capacity that can only carry commodity 2 and an edge $v'_3v'_{12}$ with $c(v'_3v'_{12}) = c^1(v'_3v'_{12}) = c^2(v'_3v'_{12}) = 1$ and preference $1 > 2$. It is easy to check that this subgadget only lets commodity 1 and commodity 2 flow in the direction we want it, and since both must pass through edge $v'_3v'_{12}$, the total flow on v_3v_{12} is at most one. From these new vertices only u_{12} and v_{12} need rankings, since only they have two incoming/outgoing edges that can forward the same commodity. Let $u_{12}v(c_1^m) >_{u_{12}} u_{12}v(c_2^m)$ and $u_3v_{12} >_{v_{12}} v_3v_{12}$ be their preferences. Later we will refer to all these edges that we added here as gadget edges. Now we can describe the non-ordinary couple-hospital edges' capacities: if c_j 's first choice is (h_j, h'_j) , then the edges $v(h_j)v(c_j^m)$ and $v(c_j^w)v(h'_j)$ can only carry commodity 2 if $j = 1, 2$ and commodity 1, if $j = 3$. Furthermore let the preferences on the $v(c_j^m)v(c_j^w)$ edges be $2 > 1$, if $j = 1, 2$ and $1 > 2$, if $j = 3$.

The flows and matchings will correspond to each other the following way: if an integer flow of commodity 1 or 2 flows through an ordinary couple's edge $e(c)$, then in the corresponding matching c is matched to their first or second choice respectively. If no flow uses $e(c)$, then c remains unassigned. If the integer flow uses edges $v(h)v(c^m)$ and $v(c^w)v(h')$ then c^m is assigned to h and c^w is assigned to h' . If $c = c_j$ $j = 1, 2$ is part of a triumvirate, then if commodity 2 flows through $v(c_j^m)v(c_j^w)$ (and then through corresponding hospitals) then c_j is matched to its first choice and if commodity 1 flows through $v(c_j^m)v(c_j^w)$ (and then through the gadget to t) then c_j is matched to its second choice (h_j, h_{j+1}) . Similarly for c_3 with the difference that commodity 1 means it is matched to

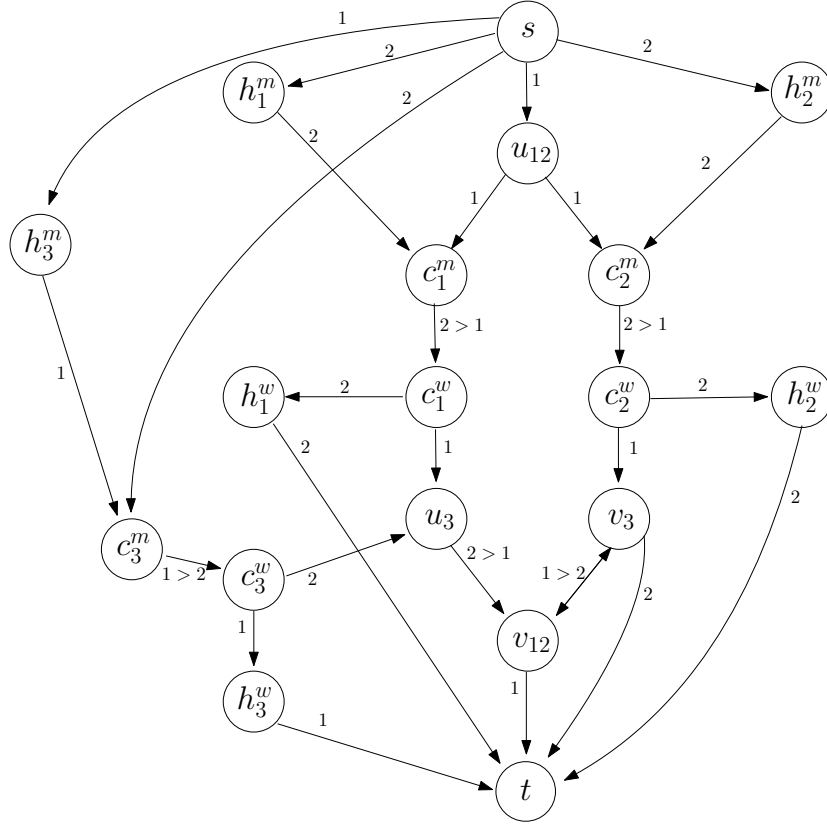


Figure 3: The gadget used in the reduction of Theorem 5.8. The h_i^m and h_i^w hospitals are c_i 's first choice hospitals. Label 1 on an edge means $c^1(e) = 1$, $c^2(e) = 0$, similarly for 2 and $1 > 2$, $2 > 1$ labels represent the edge's ranking when both commodity can travel on it.

780 its first choice and commodity 2 (which can flow through the gadget) that it is matched to its second choice.

Conversely, if an ordinary couple c is matched to hospitals (h, h') , then we send a flow of commodity 1 or 2 (depending on whether its cs first or second choice) through $s \rightarrow v(h) \rightarrow v(c^m) \rightarrow v(c^w) \rightarrow v(h') \rightarrow t$. If a non-ordinary
785 couple c_j is matched to its first choice, then we do the same, if it is matched to its second choice, then we send a flow of commodity 1 if $j = 1, 2$ and commodity 2, if $j = 3$ through the gadget. (for clarity, for c_1 the path is $s \rightarrow u_{12} \rightarrow v(c_1^m) \rightarrow v(c_1^w) \rightarrow u_3 \rightarrow v_{12} \rightarrow t$, for c_2 it is $s \rightarrow u_{12} \rightarrow v(c_2^m) \rightarrow v(c_2^w) \rightarrow u_3 \rightarrow v_{12} \rightarrow t$ and for c_3 it is $s \rightarrow v(c_3^m) \rightarrow v(c_3^w) \rightarrow u_3 \rightarrow v_{12} \rightarrow v_3 \rightarrow t$).

790 **Lemma 5.9.** *If M is a feasible matching in $\mathcal{I}$, then the constructed flow f is feasible in $\mathcal{I}'$ and vice versa.*

Proof. Let M be a feasible matching. Then every couple is assigned to at most one hospital and every hospital has at most one resident. So on each path that goes through an ordinary couples edge, there is at most one unit of flow and
795 similarly through each $sv(h)$ and $v(h)t$ edge for each man or woman-hospital. Other edges not in gadgets are only forwarding flow between these types of edges, so they aren't oversaturated either. To see the flow is feasible on the gadgets too, observe that its feasibility on the $v(c_j^m)v(c_j^w)$ edges follow from the same argument, and on the other edges it could only fail if we send more than
800 one unit of flow through su_{12} , u_3v_{12} or $v_{12}v_3$, but either would mean that two of the corresponding c_1, c_2, c_3 couples are assigned to h_1, h_2, h_3 , so at least one hospital exceeds its quota, contradiction.

Now, if f is feasible, then on each couple and ordinary hospital edge there is at most one unit of flow, so the capacity constraints of couples and man; or
805 woman-hospitals are satisfied. And since only one unit of flow can run on each of the edges su_{12} , u_3v_{12} or $v_{12}v_3$, the capacity of the non-ordinary hospitals are satisfied too. $\square$

Lemma 5.10. *M is stable if and only if f is stable.*

Proof. Suppose M is stable, but f is not, so there exists a blocking path W
810 with commodity l ($l = 1$ or 2). It can only be a path, since the set of edges on which a given commodity can travel form an acyclic graph.

Case a) W doesn't go through a gadget edge.

Case a1) W is an $s \rightarrow t$ path.

In this case W goes through an $sv(h^m)$ edge, an $e(c)$ edge and a $v(h^w)t$ edge
815 too. All of them are either undersaturated or a worse commodity goes through them, which means that the couple is either unemployed or it is at its second choice, while (h^m, h^w) is its first choice, h^m is either empty, or the man at h^m is worse for h^m and similarly for h^w . But either of the options means that (c, h^m, h^w) is a blocking coalition, contradiction.

820 *Case a2)* W is not an $s \rightarrow t$ path.

This means that it starts at a vertex $v(h^m)$ or ends at a vertex $v(h^w)$ or both. It cannot start or end at any other vertex, since there is no other vertex

from where there are two outgoing or two incoming edges that could carry the same commodity (except in the gadget). If W starts at $v(h^m)$, then there has to be a flow of the same commodity on the other edge from $v(h^m)$, that $v(h^m)$ prefers less. But this means that h^m has a resident it prefers less than c^m . Similarly if W ends at $v(h^w)$ then the resident at h^w is worse for h^w than c^w . So we get that (c, h^m, h^w) is a blocking coalition in this case too, contradiction again.

Case b) W goes through a gadget edge.

By the construction of the gadgets we basically have only 3 choices for (W, l) : The first one is $W = s \rightarrow u_{12} \rightarrow v(c_1^m) \rightarrow v(c_1^w) \rightarrow u_3 \rightarrow v_{12} \rightarrow t$ and $l = 1$, which would mean that c_1 is unassigned (because $v(c_1^m)v(c_1^w)$ has to be unsaturated, since $2 > 1$ for them), and commodity 2 does not flow through the gadget, since then $(W, 1)$ would not be blocking on u_3v_{12} . So c_3 is not assigned to (h_3, h_1) , therefore (c_1, h_1, h_2) blocks, contradiction. Here, since $u_{12}v(c_1^m) >_{u_{12}} u_{12}v(c_2^m)$ and $u_3v_{12} >_{v_{12}} v_3v_{12}$, W could start at u_{12} and end at v_{12} , but this would mean the same (here we know in addition that c_2 must be at (h_2, h_3) , but that does not change the blocking of (c_1, h_1, h_2) , since c_1 is better than c_2 at h_2). The second choice is $W = s \rightarrow v(c_3^m) \rightarrow v(c_3^w) \rightarrow u_3 \rightarrow v_{12} \rightarrow v_3 \rightarrow t$, $l = 2$. This means that c_3 is unassigned and commodity 1 does not flow through the gadget on v_3v_{12} (since commodity 2 could not block there), so c_2 is not assigned to (h_2, h_3) and therefore (c_3, h_2, h_3) blocks. The third choice is $W = s \rightarrow u_{12} \rightarrow v(c_2^m) \rightarrow v(c_2^w) \rightarrow u_3 \rightarrow v_{12} \rightarrow t$, $l = 1$. This means c_1 is not at (h_1, h_2) (since then su_{12} would be saturated) and c_2 is unassigned, so (c_1, h_1, h_2) blocks, contradiction again.

Now let us suppose we have a stable flow f , but there is a coalition (c, h, h') blocking M .

Case a) c is an ordinary couple.

Then $h = h^m$ and $h' = h^w$ for some man; and woman-hospitals respectively. Since c prefers these to its current assignment, either there is no flow through $e(c)$ or it is of commodity 2. The hospitals would hire the residents in c too, which means by the definition of the rankings, that either there is no flow through $sv(h^m)$ and $v(h^w)t$ or another type of flow goes through them that the corresponding edges prefer less, or if the same commodity goes through them, then $v(h^t)$ would prefer to send or receive the commodity on the other edge. Either would mean that there is a blocking path through these vertices (with commodity 1 if it was c 's first choice and commodity 2 otherwise), contradiction.

Case b) c is not an ordinary couple.

If h and h' are man; and woman-hospitals respectively, then using the same reasoning and that since c cannot be at its first choice, the edge $v(c_j^m)v(c_j^w)$ is either empty or has the commodity which it prefers less, we can construct a blocking walk with commodity 2 ($j = 1, 2$) or commodity 1 ($j = 3$) through $v(h)$, $e(c)$ and $v(h')$ again.

If $(c, h, h') = (c_1, h_1, h_2)$, then c_1 is unassigned and c_3 cannot be at (h_3, h_1) , so there is no flow of commodity 2 through the gadget. There may be a flow through $s \rightarrow u_{12} \rightarrow v(c_2^m) \rightarrow v(c_2^w) \rightarrow u_3 \rightarrow v_{12} \rightarrow t$, but since $u_{12}v(c_1^m) >_{u_{12}} u_{12}v(c_2^m)$ and $u_3v_{12} >_{v_{12}} v_3v_{12}$, either $W = s \rightarrow u_{12} \rightarrow v(c_1^m) \rightarrow v(c_1^w) \rightarrow u_3 \rightarrow$

870 $v_{12} \rightarrow t$ or $W = u_{12} \rightarrow v(c_1^m) \rightarrow v(c_1^w) \rightarrow u_3 \rightarrow v_{12}$ is a blocking walk with commodity 1.

If $(c, h, h') = (c_2, h_2, h_3)$, then c_2 is unassigned and c_1 cannot be at (h_1, h_2) , so there can only be a flow of commodity 2 on the gadget, but either way $W = s \rightarrow u_{12} \rightarrow v(c_2^m) \rightarrow v(c_2^w) \rightarrow u_3 \rightarrow v_{12} \rightarrow t$ blocks with commodity 1.

875 Finally if $(c, h, h') = (c_3, h_3, h_1)$, then c_3 is unassigned and c_2 isn't at (h_1, h_2) , therefore $W = s \rightarrow v(c_3^m) \rightarrow v(c_3^w) \rightarrow u_3 \rightarrow v_{12} \rightarrow v_3 \rightarrow t$ blocks with commodity 2. $\square$

So we have proven that integral stable matching in $\mathcal{I}$ and integral stable flows in $\mathcal{I}'$ correspond to each other, which proves that 2-ISMF is NP-complete indeed. $\square$

880 Using Lemma 4.3, we can prove alternate versions of Theorem 4.11 and 5.8.

Theorem 5.11. *2-ISMF is NP-complete and 3-SMF is PPAD-complete even if all capacities are 1, all commodities have the same source and terminal, the $>_v^i$ preferences are the same for all i for any $v \in V$, there are no commodity specific capacities and $\rho(v) + \delta(v) \leq 3$ for all non s, t vertices.*

885 Building on this, now we can observe that if we reduce from the above mentioned version of the SMF in Lemma 4.6 we get a 3-partite graph where each vertex in the couple's class has degree at most 3 and the other vertices have degree at most 4. So we can strengthen our results to the following:

890 **Theorem 5.12.** *The (3,4)-F-HRC problem is PPAD-complete, even if each hospital only receives applications from men or women.*

6. Conclusions

In this paper we showed that finding stable fractional solutions in many generalizations of the stable matching problem is PPAD-complete even with quite severe restrictions. These results also imply that finding a near feasible integer solution in such problems cannot be done efficiently by finding and then rounding a stable fractional solution, unless PPAD-complete problems are solvable in polynomial time. However in some cases it is still open whether these conditions can be strengthened even more. We have seen, that in FRACTIONAL HYPERGRAPH MATCHING, relaxing the conditions further would place the problem in P . However, whether the SMF is PPAD-complete if there are only two commodities is still a mystery. The same holds for (α, β) -F-HRC for some small α, β pairs with $\alpha \leq 2$ or $\beta \leq 3$.

905 Another interesting topic to research would be to search for algorithms that can find near-feasible integer solutions in polynomial-time. Of course these cannot start by finding a fractional solution, but there could be other ways of finding one. Also, it would be interesting to know that in a given problem like the ISMF, how much we need to modify the capacities in the worst case, to always obtain an instance that has an integer stable solution.

7. Acknowledgements

910 I would like to thank my supervisor Tamás Király for bringing these topics
to my attention and for all the help he gave me during my research. The
research was supported by the Momentum Program of the Hungarian Academy
of Sciences, grant number LP2021-1/2021.

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