

Approximation Algorithms for Matroidal and Cardinal Generalizations of Stable Matching

Gergely Csáji*

Tamás Király†

Yu Yokoi‡

Abstract

The Stable Marriage problem (SM), solved by the famous deferred acceptance algorithm of Gale and Shapley (GS), has many natural generalizations. If we allow ties in preferences, then the problem of finding a maximum solution becomes NP-hard, and the best known approximation ratio is 1.5 (McDermid ICALP 2009, Paluch WAOA 2011, Z. Király MATCH-UP 2012), achievable by running GS on a cleverly constructed modified instance. Another elegant generalization of SM is the matroid kernel problem introduced by Fleiner (IPCO 2001), which is solvable in polynomial time using an abstract matroidal version of GS. Our main result is a simple 1.5-approximation algorithm for the matroid kernel problem with ties. We also show that the algorithm works for several other versions of stability defined for cardinal preferences, by appropriately modifying the instance on which GS is executed. The latter results are new even for the stable marriage setting.

1 Introduction

The deferred acceptance algorithm of Gale and Shapley [8] is a quintessential example of a simple combinatorial algorithm that has wide-ranging applications, in such diverse areas as healthcare labor markets, kidney exchange planning, project allocations, and school choice mechanisms. The original stable marriage problem solved by the Gale–Shapley algorithm has been generalized in many directions, and the mathematical research in the area is still thriving, 60 years after the original paper.

The aim of the present paper is to bring together two directions in which the problem has been extended. One is the design of approximation algorithms for finding a maximum stable matching when ties are allowed in the preference lists. The other is the generalization of the stable matching problem to matroid intersection, in particular, the matroid kernel problem introduced and solved by Fleiner [6] using an abstract version of the Gale–Shapley algorithm.

We show that the best known approximation ratio of 1.5 for the stable marriage problem with ties [16] can also be achieved for the matroid kernel problem with ties. Furthermore, we can go beyond ordinal preferences with ties, and achieve the same approximation ratio for problems with cardinal preferences and various near-stability requirements. In all of these cases, our algorithms are simple variants of the Gale–Shapley algorithm applied to carefully constructed modified instances of the problem. The key observation that enables the extension of the proof to matroid kernels is an exchange property for ordered matroids that may be of independent interest.

1.1 Basic definitions

Stable marriage with ties. A *weak order* on a ground set S is an ordering of S that may contain ties; in other words, it is a partial order where being incomparable is a transitive relation. We use $\succsim$ to denote a weak order; if several weak orders are given, we use indices to distinguish them. The notation $x \sim y$ means that x is tied with y .

In the *stable marriage problem with ties and incomplete lists* (SMTI), we are given a bipartite graph $G = (U, W; E)$, and a weak order $\succsim_v$ on $\delta_G(v)$ for every vertex $v \in U \cup W$, where $\delta_G(v)$ denotes the set of

*MTA-ELTE Momentum Matroid Optimization Research Group, Department of Operations Research, Eötvös Loránd University, Budapest, Hungary. Email: csajigergely@student.elte.hu

†MTA-ELTE Egerváry Research Group, Department of Operations Research, Eötvös Loránd University, Budapest, Hungary. Email: tamas.kiraly@ttk.elte.hu

‡Principles of Informatics Research Division, National Institute of Informatics, Tokyo, Japan. Email: yokoi@nii.ac.jp

edges incident to v in G . Given a matching N in G , an edge $e = uw \in E \setminus N$ is a *blocking edge* for N if the following two conditions hold:

- $\delta_G(u) \cap N = \emptyset$ or $e \succ_u N(u)$,
- $\delta_G(w) \cap N = \emptyset$ or $e \succ_w N(w)$,

where $N(u)$ denotes the edge of N incident to u if it exists. The matching N is *stable* if no edge blocks it. The MAX-SMTI problem is to find a stable matching of maximum size. The problem is NP-hard [10], and the best known polynomial-time approximation algorithm has an approximation ratio of 1.5 [16]. It is also known that no approximation ratio better than $4/3$ is achievable assuming the Unique Games Conjecture [21].

If no ties are allowed in the preference orders, then we obtain the standard stable marriage problem, where all stable matchings have the same size, and the Gale–Shapley algorithm finds one efficiently. In fact, we can obtain a stable matching of an SMTI instance by breaking the ties arbitrarily, and running the Gale–Shapley algorithm with the resulting linear orders; however, the size of the obtained stable matching depends on how we break the ties, and it can be as small as half the optimum.

Matroid kernels. A natural way to generalize the stable marriage problem and the MAX-SMTI problem is to allow agents to have multiple partners, but have some restrictions on the possible sets of partners. There are several models in this vein: the hospitals-residents problem, the college admissions problem with common quotas, classified stable matchings etc. (see subsection 1.3 for more details).

From a theoretical point of view, a particularly elegant generalization is the matroid kernel problem defined by Fleiner [6]. We assume familiarity with the fundamental notions of matroid theory (independence, bases, fundamental circuits). We use the notation and terminology of [20, Chapter 39] for matroids, unless otherwise stated. A *weakly ordered matroid* is a triple $(S, \mathcal{I}, \succsim)$ where S is the set of elements, $\mathcal{I}$ is the family of independent sets of the matroid, and $\succsim$ is a weak order on S . Let $M_1 = (S, \mathcal{I}_1, \succsim_1)$ and $M_2 = (S, \mathcal{I}_2, \succsim_2)$ be weakly ordered matroids on the same ground set S . A common independent set $X \in \mathcal{I}_1 \cap \mathcal{I}_2$ is an (M_1, M_2) -*kernel* if for every $y \in S \setminus X$ there exists $i \in \{1, 2\}$ such that $X + y \notin \mathcal{I}_i$ and $x \succsim_i y$ for every $x \in X$ for which $X - x + y \in \mathcal{I}_i$. If there is an element $y \in S \setminus X$ for which this does not hold, then we say that y *blocks* X . The MAX-KERNEL problem is to find an (M_1, M_2) -kernel of maximum size.

If M_1 and M_2 are partition matroids, then the MAX-KERNEL problem is equivalent to MAX-SMTI. Indeed, we can construct a bipartite graph by considering the partition classes of the two matroids as vertices, and the elements of S as edges, whose two endpoints are the vertices corresponding to the two partition classes containing that element. There is a one-to-one correspondence between stable matchings of this bipartite graph and (M_1, M_2) -kernels.

To see how the matroid kernel problem is considerably more general than the stable marriage problem, observe that in the above construction we have a 1-uniform matroid for each partition class. We can replace these with arbitrary matroids, and still have a special case of the matroid kernel problem. This corresponds to a generalization of stable marriage where each agent has a matroid on the set of incident edges, and the chosen edge set is required to be independent in each of these matroids. Furthermore, we can include upper bounds on the number of edges incident to subsets of agents on both sides, provided that for each side these upper bounds are given by a polymatroid function.

Fleiner [6, 7] considered the matroid kernel problem without ties, i.e., $\succsim_1$ and $\succsim_2$ are linear orders. He showed that matroid kernels always exist and have the same size (in fact, they have the same span in both matroids). He also gave a matroidal version of the Gale–Shapley algorithm that finds a matroid kernel efficiently. In case of weak orders, kernels may have different sizes, and it is NP-hard to find a largest one, since this problem is a generalization of MAX-SMTI.

Cardinal preferences and near-stability. One way to define a weak order on a ground set S is to assign a preference value $p(x)$ to each $x \in S$, with larger value being better. Without loss of generality, we will assume that each $p(x)$ is nonnegative, and we also define $p(\emptyset) = 0$. Such cardinal preferences also allow the definition of various versions of near-stability. Intuitively, the blocking of a solution may require some quantifiable effort from the blocking agents, so we may say that an element does not block unless the improvement is at least some fixed positive Δ .

We define near-stability in the context of matroid kernels; the definitions carry over naturally to the special case of stable marriage (for the latter, similar definitions appeared in the literature under various names [1, 4, 19]). For $i \in \{1, 2\}$, let $M_i = (S, \mathcal{I}_i)$ be matroids, and for each element $x \in S$ let $p_i(x) \geq 0$ be its value in the matroid M_i . The values p_i define a weak order $\succsim_i$ on S by $x \succsim_i y \Leftrightarrow p_i(x) \geq p_i(y)$. As mentioned above, we assume that $p_1(\emptyset) = p_2(\emptyset) = 0$.

Let $\Delta > 0$. Given a common independent set X , an element $y \in S \setminus X$ is said to be Δ -min-blocking for X if there exist $x_1 \in X \cup \{\emptyset\}$ and $x_2 \in X \cup \{\emptyset\}$ such that

- $x_i = \emptyset$ if $X + y \in \mathcal{I}_i$, otherwise $X - x_i + y \in \mathcal{I}_i$, for $i \in \{1, 2\}$,
- $\min\{p_1(y) - p_1(x_1), p_2(y) - p_2(x_2)\} \geq \Delta$.

Informally, an element Δ -min-blocks X if we can achieve an improvement of Δ in both matroids, by adding it to X or by exchanging it with an element of X . The set X is Δ -min-stable if there is no Δ -min-blocking element. If all p_i values are positive and Δ is small enough, then Δ -min-stability is equivalent to being an (M_1, M_2) -kernel. Note also that in the stable marriage setting, it would make sense to have a different threshold (i.e., a different Δ value) for each agent. However, by rescaling the preference values appropriately, we can assume that all these thresholds are the same.

We will also consider two other versions of near-stability, for which 1.5-approximation can be achieved using a slightly more complicated construction. These are presented in Section 4.

1.2 Our contribution The key tool for generalizing the 1.5-approximation to matroid kernels is a result on the existence of a perfect matching of certain types of exchangeable pairs in a matroid. This is presented as Theorem 2.1 in Section 2. Due to its simplicity and apparent usefulness, it is conceivable that this result has already been discovered in other contexts. However, we did not find it in the literature, so it is certainly not as well known as it should be.

Using Theorem 2.1, we show in Section 3 that there is a 1.5-approximation algorithm for MAX-KERNEL, which consists of three steps: (1) constructing an instance of the matroid kernel problem without ties on the ground set obtained by replacing each element by 3 parallel elements, (2) running Fleiner's algorithm on the new instance, and (3) projecting the solution to the original ground set. Furthermore, given cardinal preferences and a threshold Δ , we show that the same algorithm can be used to find a 1.5-approximation for the maximum size Δ -min-stable common independent set. The running time of the algorithm is quadratic in $|S|$, and linear in case of partition matroids (which corresponds to the many-to-many stable matching problem with parallel edges allowed).

Finally, in Section 4, we show that the same general framework can be used to obtain efficient 1.5-approximation algorithms for two other natural near-stability notions: Δ -sum-stability and Δ -max-stability. These notions and their motivation are described in detail in Section 4. Informally, in addition to the standard requirement for blocking, we also require that the total improvement in the two matroids should be at least Δ in the former definition, while we require at least one of the improvements to be Δ in the latter definition. The algorithms for these problems have the same steps as above, the only difference being the construction of the new ground set and linear orders: we need $\mathcal{O}(|S|^2)$ parallel elements in case of Δ -sum-stability, and 4 parallel elements in case of Δ -max-stability. All three of our Δ -stability notions generalize SMTI, which corresponds to the case when the preference values are positive and Δ is small enough.

1.3 Related work The *stable marriage problem with ties and incomplete lists* was first studied by Iwama et al. [10], who showed the NP-hardness of MAX-SMTI. Since then, various algorithms have been proposed to improve the approximation ratio [11, 12, 13], and the current best ratio is 1.5 by a polynomial-time algorithm of McDermid [16], where the same ratio is attained by linear-time algorithms of Paluch [17, 18] and Király [14, 15]. The 1.5-approximability extends to the many-to-one matching setting [15] and the student-project allocation problem with ties [5].

The stable marriage problem also has generalizations in which constraints are imposed on the possible sets of edges. Biró et al. [3] studied the *college admissions problem with common quotas*. It is a many-to-one matching problem between students and colleges, where not only individual colleges but also certain subsets of colleges have upper quotas. Huang [9] investigated the *classified stable matching problem*, a many-to-one matching model in which each individual college has upper and lower quotas for subsets of students. In the absence of lower quotas,

it was shown that a stable matching exists in these models if constraints have laminar structure. As these models consider strict preferences, the laminar cases (without lower quotas) can be reduced to the matroid framework of Fleiner [6]. Closer to our setting, a recent work of Yokoi [22] considered a many-to-many matching model with ties and laminar constraints and presented a 1.5-approximation algorithm for the generalized MAX-SMTI. Its approximation analysis depends on the base orderability of laminar matroids and cannot extend to the general matroid setting.

Cardinal preferences in the context of stable matchings have been studied for several reasons. Pini et al. [19] analyzed manipulations consisting of falsely reporting preference values. Among other stability notions, they introduced α -stability, which is equivalent to our definition of Δ -min-stability. Anshelevich et al. [1] considered approximate stability from the point of view of social welfare. They defined various utility models, and α -stability with respect to these models. They gave price-of-anarchy bounds that depend on the value of α . Chen et al. [4] defined *local d -near-stability* and *global d -near-stability* based on swaps in the preference orders. The former can be seen as a special case of our Δ -min-stability notion with Δ being integer and the preferences being ordinal. They proved hardness of approximation results regarding the minimum value d , such that there exists a locally/globally d -nearly-stable matching that is complete or has welfare higher than some number η .

2 Existence of perfect matching of exchange edges in matroids

In this section, we present our key tool, a result on exchange properties of matroids. Let $M = (S, \mathcal{I})$ be a matroid, where S is the ground set and $\mathcal{I}$ is the family of independent sets. Recall that a base is a maximum size independent set, while a circuit is an inclusionwise minimal dependent set. The *fundamental circuit* of an element $x \in S \setminus B$ for a base B , denoted by $C_B(x)$, is the unique circuit in $B + x$. By a slight abuse of notation, we will also use $C_I(x)$ for an independent set I and an element $x \in S \setminus I$ to denote the unique circuit in $I + x$ if it exists. Any pair of circuits satisfies the following property.

PROPOSITION 2.1. (STRONG CIRCUIT AXIOM) *If C, C' are circuits, $x \in C \setminus C'$, and $y \in C \cap C'$, then there is a circuit $C'' \subseteq C \cup C'$ such that $x \in C''$ and $y \notin C''$.*

If we have a strict linear order $\succ$ given on S , then the triple $M = (S, \mathcal{I}, \succ)$ is called an *ordered matroid*. A nice property of ordered matroids is that for any weight vector $w \in \mathbb{R}^S$ which satisfies $w_x > w_y \Leftrightarrow x \succ y$, the unique maximum weight base is the same. We call this base A the *optimal base* of $(S, \mathcal{I}, \succ)$; it is characterized by the property that the worst element of $C_A(x)$ is x for any $x \in S \setminus A$. A similar statement about arbitrary circuits can be easily seen using the strong circuit axiom as follows.

LEMMA 2.1. *Let A be the optimal base of an ordered matroid $(S, \mathcal{I}, \succ)$, and let C be a circuit. Then, the worst element of C is in $S \setminus A$.*

Proof. Suppose for contradiction that there are circuits with worst element being in A , and choose such a circuit C with $|C \setminus A|$ being the smallest possible. Clearly, $C \setminus A \neq \emptyset$, since A is a base. Let $x \in C \setminus A$, and let C' be the fundamental circuit of x for A . By the optimality of A , every element $y \in C' - x$ satisfies $y \succ x$.

Let z be the worst element of C . Then $z \prec x$, so $z \notin C'$. By the strong circuit axiom, there is a circuit $C'' \subseteq C \cup C'$ such that $z \in C''$ and $x \notin C''$. This is a contradiction, because C'' also satisfies the property, and $|C'' \setminus A| < |C \setminus A|$. $\square$

We are now ready to prove the theorem that will be our main tool in proving the approximation bounds for our algorithms. As far as we know, this result on exchanges has not been observed previously in the literature. A *block matroid* is a matroid whose ground set can be partitioned into two bases.

THEOREM 2.1. *Let $M = (S, \mathcal{I}, \succ)$ be an ordered block matroid of rank r , with the property that the complement of the optimal base A is also a base, denoted by B . Then, there is a perfect matching $a_i b_i$ ($i \in [r]$) between A and B such that $a_i \succ b_i$ and $b_i \in C_B(a_i)$ for every $i \in [r]$.*

Proof. Let $G = (A, B; E)$ be the bipartite graph formed by the pairs $a \in A, b \in B$ such that $a \succ b$ and $b \in C_B(a)$. Suppose for contradiction that there exists a set $X \subseteq A$ such that $|X| > |\Gamma_G(X)|$, where $\Gamma_G(X) = \{b \in B : \exists a \in X, ab \in E\}$. Let $Y := \Gamma_G(X)$.

Notice that if C is a circuit such that $C \cap A \subseteq X$, then the worst element of C is in Y . Indeed, by Lemma 2.1, the worst element of C is in B , and this element b must be in the fundamental circuit $C_B(a)$ for some $a \in C \cap A \subseteq X$, because otherwise we could obtain a circuit in B by repeatedly removing elements in A using the strong circuit axiom, which would contradict the independence of B . Hence, $ab \in E$, and therefore $b \in Y$.

For each $x \in X$, let C_x be a circuit such that $C_x \cap A \subseteq X$, x is the worst element in $C_x \cap A$, and, subject to this, the worst element of $C_x \cap B$ is best possible (note that C_x always exists because $C_B(x)$ satisfies the first two properties). Let $y(x) \in Y$ denote the worst element of $C_x \cap B$. Since $|X| > |Y|$, there are elements $x \in X$ and $x' \in X$ such that $x \prec x'$ and $y(x) = y(x') =: y$. Notice that $x \notin C_{x'}$ because the worst element of $C_{x'} \cap A$ is x' . By applying the strong circuit axiom for C_x and $C_{x'}$, we can obtain a circuit C with the following properties:

- $C \subseteq C_x \cup C_{x'} - y$, so $C \cap A \subseteq X$, and the worst element of $C \cap B$ is better than y
- $x \in C$, so the worst element $C \cap A$ is x .

These properties contradict the choice of the circuit C_x , since C would have been a better choice. This contradiction implies that X cannot exist, so there is a perfect matching in G by Hall's theorem. $\square$

3 Matroid kernel algorithm for weakly ordered matroids

In this section, we give a simple 1.5-approximation algorithm for MAX-KERNEL. A similar algorithm was presented by Yokoi [22] for a generalization of MAX-SMTI that included laminar constraints. Her proof relied crucially on the property that the matroids induced by the laminar constraints are base orderable. In contrast, our proof works for arbitrary weakly ordered matroids.

To show the flexibility and usefulness of the algorithm, we prove the approximation ratio for the more general problem of finding a maximum size Δ -min-stable common independent set, as defined in subsection 1.1. Note that if all p_i values are positive and Δ is small enough, then Δ -min-stability is equivalent to being an (M_1, M_2) -kernel.

3.1 Description of the algorithm Let $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$ be matroids on the same ground set S . We use $C_I^1(u)$ and $C_I^2(u)$ to denote fundamental circuits in M_1 and M_2 , respectively. Let $\Delta > 0$ be a positive threshold, and let $p_i(v) \geq 0$ ($v \in S$, $i \in \{1, 2\}$) be the cardinal preferences for the two matroids. In the following, we describe the 1.5-approximation algorithm for finding the maximum size Δ -min-stable common independent set. Essentially, the algorithm creates a new instance by replacing each element by 3 parallel elements, and defines strict linear orders on the extended ground set based on the preferences on the original ground set. For the obtained ordered matroids M_1^* and M_2^* , an (M_1^*, M_2^*) -kernel A^* can be found in $\mathcal{O}(|S|^2)$ time by Fleiner's algorithm. The set A returned by the algorithm is the projection of A^* to the original ground set S .

To complete the description of the algorithm, we have to define the strict linear orders. Let the extended ground set be $S^* := \cup_{u \in S} \{x_u, y_u, z_u\}$. We define the ordered matroid $M_i^* = (S^*, \mathcal{I}_i^*, \succ_i^*)$ as follows. The elements x_u, y_u, z_u are parallel in M_i^* , that is,

$$\mathcal{I}_i^* = \{I^* \subseteq S^* : \pi(I^*) \in \mathcal{I}_i, |I^* \cap \{x_u, y_u, z_u\}| \leq 1 \ \forall u \in S\},$$

where $\pi(I^*) = \{u \in S : I^* \cap \{x_u, y_u, z_u\} \neq \emptyset\}$. To define the linear orders $\succ_1^*$ and $\succ_2^*$, we first define cardinal preferences on the extended ground set as follows.

- $p_1^*(z_u) = p_1(u)$, $p_1^*(y_u) = p_1(u) + K$, $p_1^*(x_u) = p_1(u) + K + \Delta$,
- $p_2^*(x_u) = p_2(u)$, $p_2^*(y_u) = p_2(u) + K$, $p_2^*(z_u) = p_2(u) + K + \Delta$,

where K is a number larger than any $p_i(u)$ ($u \in S$, $i \in \{1, 2\}$). The linear orders $\succ_1^*$ and $\succ_2^*$ on S^* are obtained by considering the preference orders given by p_1^* and p_2^* , and breaking the ties so that $y_u \succ_1^* x_v$ if $p_1^*(y_u) = p_1^*(x_v)$ and $y_u \succ_2^* z_v$ if $p_2^*(y_u) = p_2^*(z_v)$ for any $u, v \in S$. This completes the construction of the ordered matroids M_1^* and M_2^* .

For completeness, here we describe how Fleiner's algorithm works for our new instance M_1^* and M_2^* . The algorithm first sets R to be an empty set and repeats the following three steps: set X to be the optimal base of M_1^* with the ground set restricted to $S^* \setminus R$, set Y to be the optimal base of M_2^* with the ground set restricted to X , and update R with $R \cup (X \setminus Y)$. The repetition stops if $Y = X$, and Y is returned, which is an (M_1^*, M_2^*) -kernel (see [2] for this version of the description).

LEMMA 3.1. *The output of our algorithm is a Δ -min-stable common independent set of M_1 and M_2 .*

Proof. Let $A = \pi(A^*)$ be the output of the algorithm, where A^* is the (M_1^*, M_2^*) -kernel given by Fleiner's algorithm. It is clear from the definition that $A \in \mathcal{I}_1 \cap \mathcal{I}_2$. Suppose for contradiction that there exists $u \in S \setminus A$ that Δ -min-blocks A ; we claim that y_u blocks A^* . Indeed, if $A + u \in \mathcal{I}_i$, then $A^* + y_u \in \mathcal{I}_i^*$, and if $p_i(u) \geq p_i(v) + \Delta$ for some $v \in C_A^i(u)$, then $v^* := \{x_v, y_v, z_v\} \cap A^*$ satisfies $y_u \succ_i^* v^*$ and belongs to the fundamental circuit of y_u for A^* in M_i^* . $\square$

3.2 Proof of 1.5-approximation

THEOREM 3.1. *The approximation ratio of the above algorithm is at most 1.5.*

Proof. Let $A = \pi(A^*)$ be the output of the algorithm, where A^* is an (M_1^*, M_2^*) -kernel, and let B be a largest Δ -min-stable common independent set of M_1 and M_2 . Suppose for contradiction that $|B| > 1.5|A|$. Let B_i be a subset of $B \setminus A$ such that $A \cup B_i \in \mathcal{I}_i$ and $|A \cup B_i| = |B|$ for each $i \in \{1, 2\}$. The sets B_1 and B_2 are disjoint because A^* is an inclusionwise maximal common independent set of M_1^* and M_2^* . In the following, we say that an element $u \in A$ is of *type* x (respectively y, z) if $\{x_u, y_u, z_u\} \cap A^* = x_u$ (respectively y_u, z_u).

LEMMA 3.2. *Let $i \in \{1, 2\}$. There is a matching N_i of size $|B_{3-i}|$ between $A \setminus B$ and B_{3-i} such that the following hold for every $uv \in N_i$, where $u \in A$ and $v \in B$:*

1. u is of type x or y if $i = 1$, and of type y or z if $i = 2$
2. $p_i(u) \geq p_i(v)$, and in particular $p_i(u) \geq p_i(v) + \Delta$ if u is of type y
3. either $v \in C_B^i(u)$ or $B + u \in \mathcal{I}_i$.

Proof. Let $M' = (S', \mathcal{I}')$ be the matroid obtained from M_i by deleting $S \setminus (A \cup B)$, contracting $(A \cap B) \cup B_i$, and truncating to the size of $A \setminus B$. That is, $S' = (A \setminus B) \cup (B \setminus (A \cup B_i))$ and $\mathcal{I}' = \{I \subseteq S' : I \subseteq A \cup B, I \cup (A \cap B) \cup B_i \in \mathcal{I}_i, |I| \leq |A \setminus B|\}$. In M' , the sets $A' := A \setminus B$ and $B' := B \setminus (A \cup B_i)$ are bases that are complements of each other. We define a strict preference order $\succ'$ on S' in the following way. The elements of $B \setminus (A \cup B_i \cup B_{3-i})$ are worst (in arbitrary order). On the remaining elements, i.e., on the elements of $(A \setminus B) \cup B_{3-i}$, we define the preferences based on the strict preferences $\succ_i^*$ on S^* . To do this, we assign an element $u^* \in S^*$ to each $u \in (A \setminus B) \cup B_{3-i}$ as follows. Let $u^* = \{x_u, y_u, z_u\} \cap A^*$ if $u \in A \setminus B$, let $u^* = x_u$ if $i = 1$ and $u \in B_2$, and let $u^* = z_u$ if $i = 2$ and $u \in B_1$. We then let $u \succ' v$ if and only if $u^* \succ_i^* v^*$. In the ordered matroid $M' = (S', \mathcal{I}', \succ')$, A' is an optimal base. Indeed, v is the worst element of $C_{A'}(v)$ for every $v \in B'$. It is clear for the elements in $B' \setminus B_{3-i}$ by the definition of $\succ'$. As for each $v \in B_{3-i}$, since $A^* + v^* \in \mathcal{I}_{3-i}^*$ holds and A^* is an (M_1^*, M_2^*) -kernel, v^* must be the worst element of its fundamental circuit for A^* . By Theorem 2.1, there is a perfect matching N' between A' and B' such that $u \succ' v$ and $v \in C_{B'}^i(u)$ for every $uv \in N'$, where $u \in A'$ and $v \in B'$.

Let N_i be the subset of N' induced by $A \cup B_{3-i}$. Then $|N_i| = |B_{3-i}|$, and the first two properties of the lemma are satisfied for every $uv \in N_i$, because $u \succ' v$. We now show that for every $uv \in N_i$, either $v \in C_B^i(u)$ or $B + u \in \mathcal{I}_i$. Since $v \in C_{B'}^i(u)$, v is in the fundamental circuit of u for B' in the matroid obtained by truncating M_i to the size of $A \setminus B$. This means that it is either in the fundamental circuit also in M_i , or $B + u$ is independent in M_i , as required. $\square$

We are now ready to prove the theorem by getting a contradiction. Since $|B| > 1.5|A|$ implies $|N_i| = |B_{3-i}| > |A \setminus B|/2$ ($i \in \{1, 2\}$), there is an element $u \in A \setminus B$ that is covered by both N_1 and N_2 . Let $uv_1 \in N_1$, $uv_2 \in N_2$. Since the first two properties hold for $i \in \{1, 2\}$, u must be of type y , and $p_i(u) \geq p_i(v_i) + \Delta$ for $i \in \{1, 2\}$. But this means that u is a Δ -min-blocking element for B because of the third property, a contradiction. $\square$

4 Extensions to other variants of Δ -stability

As a motivation, let us consider the special case of partition matroids, which corresponds to a two-sided matching problem with cardinal preferences. The motivation for Δ -min-stability is that blocking may require some effort from the agents, so we say that a pair does not block unless they *both* achieve a value increase of Δ . However, there are other natural requirements that we can associate to a given positive threshold Δ . In some applications, blocking can be viewed as a combined effort of a pair, so we may require that the *sum* of their increase in value

should be at least Δ . In other applications, an extra effort may be required by the agent who initiates the blocking, so we might say that a pair does not block unless *one of them* achieves a value increase of Δ .

In this section, we introduce the precise definitions for the above-mentioned variants, called Δ -sum-stability and Δ -max-stability. For each of these stability concepts, finding a largest solution is NP-hard as it is a generalization of MAX-SMTI. We show that, by a suitable modification of the construction of the extended ground set and the ordered matroids M_1^* and M_2^* , we obtain 1.5-approximation algorithms for these variants of Δ -stability, too.

4.1 Δ -sum-stability Recall that we are given two matroids $M_1 = (S, \mathcal{I}_1)$ and $M_2 = (S, \mathcal{I}_2)$, as well as preferences defined by nonnegative values $p_i(v)$ for $v \in S$, $i \in \{1, 2\}$. Let Δ be a positive threshold.

DEFINITION 4.1. *Let X be a common independent set of M_1 and M_2 . An element $u \in S \setminus X$ is Δ -sum-blocking for X if the following hold:*

- *either $X + u \in \mathcal{I}_1$ or there is an element $v_1 \in X$ such that $p_1(u) > p_1(v_1)$ and $v_1 \in C_X^1(u)$,*
- *either $X + u \in \mathcal{I}_2$ or there is an element $v_2 \in X$ such that $p_2(u) > p_2(v_2)$ and $v_2 \in C_X^2(u)$,*
- *$p_1(u) - p_1(v_1) + p_2(u) - p_2(v_2) \geq \Delta$, where we take $v_i = \emptyset$ if $X + u \in \mathcal{I}_i$.*

A common independent set X is Δ -sum-stable if there is no Δ -sum-blocking element for X .

Our aim is to give an efficient 1.5-approximation algorithm for the problem of finding the largest Δ -sum-stable common independent set. As in the case of Δ -min-stability, we create an instance of the matroid kernel problem by adding parallel elements, and by defining strict linear orders on the extended ground set based on the preferences. However, the number of parallel elements will depend on the possible differences in the preference values.

Let $0 < d_1 < d_2 < \dots < d_k < \Delta$ be the set of numbers strictly between 0 and Δ that can be obtained in the form $p_1(u) - p_1(v)$, $p_2(u) - p_2(v)$, $\Delta - p_1(u) + p_1(v)$, or $\Delta - p_2(u) + p_2(v)$ for some $u, v \in S \cup \{\emptyset\}$. Furthermore, let $d_0 = 0$. Clearly, $k \leq \mathcal{O}(|S|^2)$. It is also easy to observe that $d_{k-\ell+1} = \Delta - d_\ell$ for $1 \leq \ell \leq k$ by the symmetry over Δ in the definition.

For each element $u \in S$, we make $k + 2$ parallel copies of u . Their set is denoted by $X_u^* := \{x_0(u), x_1(u), \dots, x_{k+1}(u)\}$. Let $S^* := \cup_{u \in S} X_u^*$, and let the resulting two matroids on S^* be M_1^* and M_2^* . The families of independent sets are given by

$$\mathcal{I}_i^* = \{I^* \subseteq S^* : \pi(I^*) \in \mathcal{I}_i, |I^* \cap X_u^*| \leq 1 \ \forall u \in S\},$$

where $\pi(I^*) = \{u \in S : I^* \cap X_u^* \neq \emptyset\}$.

Next, we define the linear orders $\succ_1^*$ and $\succ_2^*$. We first define cardinal preferences on the extended ground set as follows.

- $p_1^*(x_{k+1}(u)) = p_1(u)$, $p_1^*(x_\ell(u)) = p_1(u) + K + \Delta - d_\ell$ for $0 \leq \ell \leq k$,
- $p_2^*(x_0(u)) = p_2(u)$, $p_2^*(x_\ell(u)) = p_2(u) + K + \Delta - d_{k-\ell+1}$ for $1 \leq \ell \leq k + 1$,

where K is a number larger than any $p_i(u)$ ($u \in S$, $i \in \{1, 2\}$). The linear orders $\succ_1^*$ and $\succ_2^*$ on S^* are obtained by considering the preference orders given by p_1^* and p_2^* , and breaking the ties according to the following rule:

- If $p_1^*(x_j(u)) = p_1^*(x_\ell(v))$ and $j < \ell$, then $x_j(u) \prec_1^* x_\ell(v)$,
- If $p_2^*(x_j(u)) = p_2^*(x_\ell(v))$ and $j < \ell$, then $x_j(u) \succ_2^* x_\ell(v)$.

This completes the construction of the ordered matroids M_1^* and M_2^* .

THEOREM 4.1. *Let A^* be an (M_1^*, M_2^*) -kernel, and let $A = \pi(A^*)$. Then A is a Δ -sum-stable common independent set of M_1 and M_2 .*

Proof. It is clear from the definition that A is a common independent set. Suppose there is a Δ -sum-blocking element $u \in S$. If either $A + u \in \mathcal{I}_1$ or $A + u \in \mathcal{I}_2$, then $x_{k+1}(u)$ or $x_0(u)$ blocks A^* respectively, a contradiction. Otherwise let v_1, v_2 be as in Definition 4.1. Let $v_i^* = A^* \cap X_{v_i}^*$ ($i = 1, 2$).

As u is Δ -sum blocking, we have $p_1(u) - p_1(v_1) > 0$, $p_2(u) - p_2(v_2) > 0$, and $p_1(u) - p_1(v_1) + p_2(u) - p_2(v_2) \geq \Delta$. If $p_1(u) - p_1(v_1) \geq \Delta$, then we claim that $x_k(u)$ blocks A^* . Indeed, on one hand, $p_1(u) - d_k > p_1(u) - \Delta \geq p_1(v_1) = p_1(v_1) - d_0$, so $x_k(u) \succ_1^* x_0(v_1) \succeq_1^* v_1^*$. On the other hand, $p_2(u) - p_2(v_2) > 0$ implies $p_2(u) - d_1 \geq p_2(v_2)$, so $x_k(u) \succ_2^* x_{k+1}(v_2) \succeq_2^* v_2^*$.

A similar argument shows that if $p_2(u) - p_2(v_2) \geq \Delta$, then $x_1(u)$ blocks A^* . Now consider the case when both are smaller than Δ . We have $p_1(u) - p_1(v_1) = d_j$, $p_2(u) - p_2(v_2) = d_\ell$ for some $0 < j, \ell \leq k$ such that $d_j + d_\ell \geq \Delta$.

We claim that $x_j(u)$ blocks A^* . First, $p_1(u) - d_j = p_1(v_1) = p_1(v_1) - d_0$, so $x_j(u) \succ_1^* x_0(v_1) \succeq_1^* v_1^*$. Second, $p_2(u) - d_{k-j+1} = p_2(u) + d_j - \Delta \geq p_2(u) - d_\ell = p_2(v_2) = p_2(v_2) - d_0$, so $x_j(u) \succ_2^* x_{k+1}(v_1) \succeq_2^* v_2^*$.

Since $A^* - v_i^* + x_j(u) \in \mathcal{I}_i^*$ for $i = 1, 2$, this means that $x_j(u)$ blocks A^* , a contradiction. $\square$

THEOREM 4.2. *Let A^* be an (M_1^*, M_2^*) -kernel, let $A = \pi(A^*)$, and let B be a maximum size Δ -sum-stable common independent set for M_1 and M_2 . Then $|B| \leq 1.5|A|$.*

Proof. Suppose for contradiction that $|B| > 1.5|A|$. We use a similar construction as in the proof of Theorem 3.1. Let B_i be a subset of $B \setminus A$ such that $A \cup B_i \in \mathcal{I}_i$ and $|A \cup B_i| = |B|$ for $i \in \{1, 2\}$. The sets B_1 and B_2 are disjoint because A^* is an inclusionwise maximal common independent set of M_1^* and M_2^* .

The following lemma is analogous to Lemma 3.2. For $u \in A$, let $u^* := A^* \cap X_u^*$.

LEMMA 4.1. *Let $i \in \{1, 2\}$. There is a matching N_i of size $|B_{3-i}|$ between $A \setminus B$ and B_{3-i} such that the following hold for every $uv \in N_i$, where $u \in A$ and $v \in B$:*

1. $u^* = x_j(u)$ for some $j \leq k$ if $i = 1$, and for some $j \geq 1$ if $i = 2$,
2. $p_i(u) \geq p_i(v) + d_j$ if $i = 1$, and $p_i(u) \geq p_i(v) + d_{k-j+1}$ if $i = 2$,
3. either $v \in C_{B_i}^i(u)$ or $B + u \in \mathcal{I}_i$.

Proof. Let $M' = (S', \mathcal{I}')$ be the same matroid as in the proof of Lemma 3.2, and let $A' := A \setminus B$, $B' := B \setminus (A \cup B_i)$. We define a strict preference order $\succ'$ on S' in the following way. The elements of $B \setminus (A \cup B_i \cup B_{3-i})$ are worst (in arbitrary order). On the remaining elements, i.e., on the elements of $(A \setminus B) \cup B_{3-i}$, we define the preferences based on the strict preferences $\succ_i^*$ on S^* . Let $v^* = x_0(v)$ if $i = 1$ and $v \in B_2$, and let $v^* = x_{k+1}(v)$ if $i = 2$ and $v \in B_1$. Let $u \succ' v$ if and only if $u^* \succ_i^* v^*$. As in the proof of Lemma 3.2, A' is an optimal base in the ordered matroid $M' = (S', \mathcal{I}', \succ')$. By Theorem 2.1, there is a perfect matching N' between A' and B' such that $u \succ' v$ and $v \in C_{B'}^i(u)$ for every $uv \in N'$, where $u \in A'$ and $v \in B'$.

Let N_i be the subset of N' induced by $A \cup B_{3-i}$. Consider $uv \in N_i$, $u \in A'$, $v \in B_{3-i}$. The first property in the lemma holds because $x_{k+1}(u) \prec_1^* x_0(v)$ and $x_0(u) \prec_2^* x_{k+1}(v)$ by the definitions of $\succ_1^*$ and $\succ_2^*$. To see the second property in the case $i = 1$, observe that $u \succ' v$ implies $x_j(u) \succ_1^* x_0(v)$, thus $p_1(u) - d_j \geq p_1(v)$. Similarly, in the case $i = 2$, $u \succ' v$ implies $x_j(u) \succ_2^* x_{k+1}(v)$, thus $p_1(u) - d_{k-j+1} \geq p_1(v)$.

The third property follows similarly to the proof of Lemma 3.2. $\square$

Since $|B| > 1.5|A|$, there is an element $u \in A \setminus B$ that is covered by both N_1 and N_2 . Let v_1, v_2 be u 's partners in N_1 and N_2 respectively. By the first property of the lemma, $u^* = x_j(u)$ for some $j \in \{1, \dots, k\}$. By the second property, $p_1(u) \geq p_1(v_1) + d_j$ and $p_2(u) \geq p_2(v_2) + d_{k-j+1}$.

Using that $d_j > 0$, $d_{k-j+1} > 0$, $d_j + d_{k-j+1} = \Delta$, and the fact that $v_i \in C_{B_i}^i(u)$ or $B + u \in \mathcal{I}_i$ for $i = 1, 2$, we get that the element u is Δ -sum-blocking for B , a contradiction. $\square$

4.2 Δ -max-stability The proofs in this section are very similar to those for Δ -sum-stability, so we skip some details that are identical. Let Δ be a positive threshold.

DEFINITION 4.2. *Let X be a common independent set of M_1 and M_2 . An element $u \in S \setminus X$ is Δ -max-blocking for X if the following hold:*

- either $X + u \in \mathcal{I}_1$ or there is an element $v_1 \in X$ such that $p_1(u) > p_1(v_1)$ and $v_1 \in C_X^1(u)$,

- either $X + u \in \mathcal{I}_2$ or there is an element $v_2 \in X$ such that $p_2(u) > p_2(v_2)$ and $v_2 \in C_X^2(u)$,
- $\max\{p_1(u) - p_1(v_1), p_2(u) - p_2(v_2)\} \geq \Delta$, where we take $v_i = \emptyset$ if $X + u \in \mathcal{I}_i$.

A common independent set X is Δ -max-stable if there is no Δ -max-blocking element for X .

In order to get a 1.5-approximation, we create a matroid kernel instance by taking four parallel elements of each element $u \in S$, denoted by $X_u^* := \{x_0(u), x_1(u), x_2(u), x_3(u)\}$. Let $S^* = \cup_{u \in S} X_u^*$. As in the previous section, the families of independent sets are given by

$$\mathcal{I}_i^* = \{I^* \subseteq S^* : \pi(I^*) \in \mathcal{I}, |I^* \cap X_u^*| \leq 1 \forall u \in S\},$$

where $\pi(I^*) = \{u \in S : I^* \cap X_u^* \neq \emptyset\}$. We introduce the following cardinal preferences on S^* :

- $p_1^*(x_3(u)) = p_1(u)$, $p_1^*(x_2(u)) = p_1(u) + K$, $p_1^*(x_1(u)) = p_1^*(x_0(u)) = p_1(u) + K + \Delta$,
- $p_2^*(x_0(u)) = p_2(u)$, $p_2^*(x_1(u)) = p_2(u) + K$, $p_2^*(x_2(u)) = p_2^*(x_3(u)) = p_2(u) + K + \Delta$,

where K is larger than any $p_i(u)$ ($u \in S, i \in \{1, 2\}$). The linear orders $\succ_1^*$ and $\succ_2^*$ on S^* are obtained by considering the preference orders given by p_1^* and p_2^* , and breaking the ties according to the following rule:

- If $p_1^*(x_j(u)) = p_1^*(x_\ell(v))$ and $j < \ell$, then $x_j(u) \prec_1^* x_\ell(v)$, except for $x_0(u) \succ_1^* x_1(v)$
- If $p_2^*(x_j(u)) = p_2^*(x_\ell(v))$ and $j < \ell$, then $x_j(u) \succ_2^* x_\ell(v)$, except for $x_2(u) \prec_2^* x_3(v)$

This completes the construction of the ordered matroids M_1^* and M_2^* .

THEOREM 4.3. *Let A^* be an (M_1^*, M_2^*) -kernel, and let $A = \pi(A^*)$. Then A is a Δ -max-stable common independent set of M_1 and M_2 .*

Proof. It is clear from the definition that A is a common independent set. Suppose there is a Δ -max-blocking element $u \in S$. If either $A + u \in \mathcal{I}_1$ or $A + u \in \mathcal{I}_2$, then $x_3(u)$ or $x_0(u)$ blocks A^* respectively. Otherwise let v_1, v_2 be as in Definition 4.2. Let $v_i^* = A^* \cap X_{v_i}^*$ ($i = 1, 2$).

As u is Δ -max-blocking, we have $p_1(u) - p_1(v_1) > 0$, $p_2(u) - p_2(v_2) > 0$, and $\max\{p_1(u) - p_1(v_1), p_2(u) - p_2(v_2)\} \geq \Delta$. By symmetry, we may assume that $p_1(u) - p_1(v_1) \geq \Delta$.

We claim that $x_2(u)$ blocks A^* . First, $p_1(u) \geq p_1(v_1) + \Delta$, so $p_1^*(x_2(u)) \geq p_1^*(v_1^*)$, and equality may hold only when $v_1^* \in \{x_0(v_1), x_1(v_1)\}$. By the tie-breaking rule, we have $x_2(u) \succ_1^* v_1^*$. Second, $p_2(u) > p_2(v_2)$, so $p_2^*(x_2(u)) > p_2^*(v_2^*)$, which means that $x_2(u) \succ_2^* v_2^*$.

Since $A^* - v_i^* + x_2(u) \in \mathcal{I}_i^*$ for $i = 1, 2$, this means that $x_2(u)$ blocks A^* , a contradiction. $\square$

THEOREM 4.4. *Let A^* be an (M_1^*, M_2^*) -kernel, let $A = \pi(A^*)$, and let B be a maximum size Δ -max-stable common independent set for M_1 and M_2 . Then $|B| \leq 1.5|A|$.*

Proof. Suppose for contradiction that $|B| > 1.5|A|$. Let B_i be a subset of $B \setminus A$ such that $A \cup B_i \in \mathcal{I}_i$ and $|A \cup B_i| = |B|$ for $i \in \{1, 2\}$. The sets B_1 and B_2 are disjoint because A^* is an inclusionwise maximal common independent set of M_1^* and M_2^* . For $u \in A$, we use the notation $u^* := A^* \cap X_u^*$.

LEMMA 4.2. *Let $i \in \{1, 2\}$. There is a matching N_i of size $|B_{3-i}|$ between $A \setminus B$ and B_{3-i} such that the following hold for every $uv \in N_i$, where $u \in A$ and $v \in B$:*

1. $u^* = x_j(u)$ for some $j \leq 2$ if $i = 1$, and for some $j \geq 1$ if $i = 2$,
2. $p_i(u) > p_i(v)$ if either $i = j = 1$ or $i = j = 2$,
3. $p_i(u) \geq p_i(v) + \Delta$ if either $i = 1$ and $j = 2$, or $i = 2$ and $j = 1$,
4. either $v \in C_B^i(u)$ or $B + u \in \mathcal{I}_i$.

Proof. Let $M' = (S', \mathcal{I}')$ be the same matroid as in the proof of Lemma 3.2, and let $A' := A \setminus B$, $B' := B \setminus (A \cup B_i)$. We define a strict preference order $\succ'$ on S' in the following way. The elements of $B \setminus (A \cup B_i \cup B_{3-i})$ are worst (in arbitrary order). On the remaining elements, i.e., on the elements of $(A \setminus B) \cup B_{3-i}$, we define the preferences based on the strict preferences $\succ_i^*$ on S^* . Let $v^* = x_0(v)$ if $i = 1$ and $v \in B_2$, and let $v^* = x_3(v)$ if $i = 2$ and $v \in B_1$. Let $u \succ' v$ if and only if $u^* \succ_i^* v^*$. With these preferences, A' is an optimal base in the ordered matroid $M' = (S', \mathcal{I}', \succ')$. By Theorem 2.1, there is a perfect matching N' between A' and B' such that $u \succ' v$ and $v \in C_{B'}'(u)$ for every $uv \in N'$, where $u \in A'$ and $v \in B'$.

Let N_i be the subset of N' induced by $A \cup B_{3-i}$. Consider $uv \in N_i$, $u \in A'$, $v \in B_{3-i}$. The first property in the lemma holds because $x_3(u) \prec_1^* x_0(v)$ and $x_0(u) \prec_2^* x_3(v)$. To see the second and third properties in the case $i = 1$, observe that $u \succ' v$ implies $x_j(u) \succ_1^* x_0(v)$. By the tiebreaking rule, this implies $p_1(u) > p_1(v)$ if $j = 1$, and $p_1(u) \geq p_1(v) + \Delta$ if $j = 2$. Similarly, in the case $i = 2$, we get $p_2(u) > p_2(v)$ if $j = 2$, and $p_2(u) \geq p_2(v) + \Delta$ if $j = 1$. The fourth property follows similarly to the proof of Lemma 3.2. $\square$

Since $|B| > 1.5|A|$, there is an element $u \in A \setminus B$ that is covered by both N_1 and N_2 . Let v_1, v_2 be u 's partners in N_1 and N_2 respectively. By the first property of the lemma, $u^* = x_j(u)$ for some $j \in \{1, 2\}$. If $j = 1$, then the second and third properties imply that $p_1(u) > p_1(v_1)$ and $p_2(u) \geq p_2(v_2) + \Delta$. If $j = 2$, then we get $p_1(u) \geq p_1(v_1) + \Delta$ and $p_2(u) > p_2(v_2)$.

By the fourth property of Lemma 4.2, we obtain that the element u is Δ -sum-blocking for B , a contradiction. $\square$

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