

Towards a Better Understanding of the Characteristics of Fractal Networks

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1 Introduction

The fractal nature of complex networks has received a great deal of research interest in the last two decades. Similarly to geometric fractals, the fractality of networks can also be defined with the so-called box-covering method. A network is called fractal, if the minimum number of boxes needed to cover the entire network follows a power-law relation with the size of the boxes: $N_B(l_B) \sim l_B^{-d_B}$ [1].

The fractality of networks has been associated with various network properties throughout the years, for example, disassortativity [2], repulsion between hubs [3], long-range-repulsive correlation [4, 5], an underlying tree structure [6], the correlation between the betweenness centrality and the degree of nodes [7], small edge betweenness centralities [8]. On the other hand, these assertions are usually based on tailor-made network models and on a small number of real networks, hence their ubiquity is often disputed [4, 8, 9]. In this contribution, we systematically review the previous results about the relationship between various network characteristics and fractality. Moreover, we perform a comprehensive analysis of these relations on network models (including newly introduced models) and a large number of real-world networks collected from various network repositories.

2 Data and Models

For the evaluation, we consider three popular fractal network models: Song-Havlin-Makse model [3], hub attraction dynamical growth model [9], and (u, v) -flower [10]. Furthermore, we also introduce two novel models, the repulsion-based fractal (RBF) model and the mixture model [11]. The RBF model uses a recursive approach to produce a network with a repulsive connection among a group of nodes within a certain degree range (driven by a user-defined parameter). The special feature of the model is that it generates fractal networks regardless of the choice of parameters, suggesting that repulsion between hubs is not a common feature of all fractal networks. The mixture model demonstrates the transition from fractal to non-fractal network, as Fig. 1(a) illustrates. It starts with a pure fractal lattice graph and reduces the distances in the network based on the preferential attachment mechanism driven by a p probabilistic parameter.

Furthermore, we study more than 250 real-world networks originating from six domains: brain, metabolic, cheminformatics, food, infrastructural, and social. Even though they describe very different interactions, the majority show fractal characteristics.

3 Results

Dissortative degree mixing has been observed in several real-world and synthetic fractal networks, thus it has become widely accepted that disassortativity is the main factor behind fractality [2]. However, later it has been verified that this assertion fails to be universal [9]. Our mixture model also verifies the absence of causal relation of the two properties. It not only generates instances for assortative fractal networks, but we can observe a positive connection between fractality and assortativity. This phenomenon is well-demonstrated by Fig. 1(b).

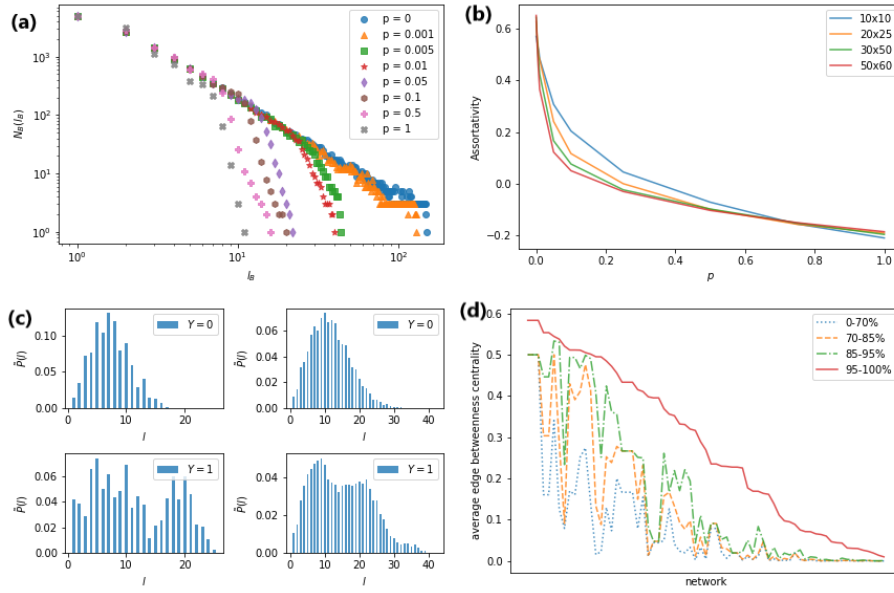


Fig. 1. (a) Transition of fractality of the mixture model, with different p parameters. (b) The change in assortativity as a function of p in the case of the mixture model for different network sizes. (c) Distribution of hub distances for the RBF model. The two columns correspond to different network sizes, while parameter Y is responsible for the creation of repulsion among a certain group of nodes. With $Y = 0$ the repulsion exists among small degree nodes, with $Y = 1$ it exists among hubs. (d) Average edge betweenness centralities of the RBF model. The edges are divided into four categories according to their betweenness centralities by the following percentile cut points: 95, 85, 70. The average betweenness centrality of the corresponding subgroups of edges for each network is illustrated with a line plot. On the x -axis, the networks are ordered by the average edge betweenness centrality. The figure shows that there are a large number of networks, for which the distribution of the betweenness centralities has a wide tail, meaning that they have a relatively large number of edges with high betweenness centrality.

Another study suggests that the emergence of fractal scaling is due to the strong repulsion between large degree nodes [3]. This statement has already been conflicted with another model, which can generate fractal networks with hub attraction behav-

ior [9]. Our RBF model also demonstrates that in fractal networks, it is not necessarily the hubs that are in a repulsive connection.

Long-range degree anticorrelation has also been associated with fractality [4, 5]. This property has been investigated on the one hand, by fluctuation analysis [5], on the other hand, by considering the distribution of the hub distances [4]. We apply both methods in our examination, together with a third approach, which generalizes neighbour connectivity to m -chain neighbours [12]. We find that long-range anticorrelation captured by fluctuation analysis and neighbour connectivity does not appear universally in every fractal network. However, as Fig. 1(c) illustrates, the hub distances indeed have a wide range for the RBF model. Hubs are often far away from each other, even when there is no direct repulsion between them.

There are several other characteristics whose possible relations to fractality have been investigated. For instance, it has been observed, that in fractal networks the correlation between the degree of nodes and their betweenness centrality is much smaller than in non-fractal ones [7]. We find, however, that the degree of correlation is independent of fractality. Another assertion is related to edge betweenness centrality, namely, that even a small amount of edges with high betweenness destroy the fractal nature of a network [8]. However, as the RBF model demonstrates a fractal network can have a large number of edges with high betweenness centrality (see Fig. 1(d)).

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