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Simple systems in 3D noncommutative quantum mechanics

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Abstract: In this paper, we examine energy spectra of two well-known quantum systems in standard quantum mechanics (QM) affected by introducing a noncommutative (NC) structure into the systems' configuration space. NC structure blurs the configuration space at short distances and the parameter in charge of describing the extent of this blur is λ . In the present case, we investigate the bound states of electron in the NC Coulomb potential and NC 3D isotropic harmonic oscillator. Since complete analytic result for the latter potential is unattainable, the radial WKB approximation is adopted. We calculate the energy of electron in the NC Coulomb potential of arbitrary-order and energy of the NC 3D isotropic harmonic oscillator of the second order in λ .

Keywords: noncommutative quantum mechanics, Planck's scale physics, radial WKB approximation

1 Introduction

NC models became popular during the 1980s in the connection with the endeavor to develop the Theory of Everything, where the quantization of gravity is required. If we attempt to incorporate general relativity and quantum mechanics, we unavoidable reach the deduction that two points of space separated by a distance at the Planck's scale ($\lambda \approx 10^{-35}$ m) cannot be distinguished. A lot of success was reached in this field of the physics, however, the complete non-commutative quantum field theory is still under construction.

In this work, we test the consequences of the NC geometry in the energy spectrum of two quantum systems: the bound states of electron in the Coulomb potential and the 3D isotropic harmonic oscillator. The energy levels of the former have already been calculated exactly, whereas the latter remained an uncharted area of physics until now.

This contribution is organized as follows. In section 2, we make acquaintance with the basic techniques of our tools: the radial WKB

(Wentzel–Kramers–Brillouin) method and the mathematics in NC geometry. We present the exact known result of the discrete energy spectrum in the hydrogen atom considering NC space. Due to the difficulty of calculations for the NC 3D isotropic harmonic oscillator, we utilized the radial WKB approximation in sections 3 and 4. Both of the potentials have been investigated by this method. In section 3, we recovered for the NC Coulomb potential exactly the same modified energy spectrum which had been derived already in [1]. In section 4, we calculated for the NC 3D isotropic harmonic oscillator the first non-zero correction to the energy levels.

2 Preliminaries

In this section, we provide some indispensable preliminaries for what follows in further sections. To keep this report as short as possible, we concentrate on the notions and expressions which will be directly used. A more thorough review of the topics of standard QM and NCQM can be found in [1, 2].

2.1 Review of standard QM

Our starting point, the time-independent Schrödinger equation, looks like

$$\left[-\frac{\hbar^2}{2m} \Delta + V(r) \right] \Psi(\mathbf{r}) = E \Psi(\mathbf{r}). \quad (1)$$

Both the Coulomb potential $V_C(r) = -\frac{1}{4\pi\epsilon_0} \frac{e^2}{r}$ and the 3D isotropic harmonic potential $V_H(r) = \frac{1}{2} m \omega^2 r^2$ are rotationally invariant and thus are the respective Hamiltonian operators. Hence, the wave function $\Psi(\mathbf{r})$ is reasonably sought in a separated form

$$\Psi(\mathbf{r}) = R_l(r) Y_{lm}(\vartheta, \varphi), \quad (2)$$

where l is the quantum number corresponding to the angular momentum, m is the quantum number corresponding to the third component of angular momentum and $Y_{lm}(\vartheta, \varphi)$ is the standard spherical harmonic function. The r -dependent function $R_l(r)$ satisfies the

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radial Schrödinger equation¹

$$0 = R''(r) + \frac{2}{r}R'(r) + \frac{2m}{\hbar^2} \left(E - V(r) - \frac{\hbar^2}{2m} \frac{l(l+1)}{r^2} \right) R(r). \quad (3)$$

In the case of the Coulomb potential V_C , the equation of the radial part is

$$R''(\rho) + \frac{2}{\rho}R'(\rho) + \left(-\varepsilon + \frac{2}{\rho} - \frac{l(l+1)}{\rho^2} \right) R(\rho) = 0, \quad (4)$$

where $\varepsilon = \frac{E}{-\frac{1}{2}mc^2\alpha^2}$ is the dimensionless energy, $\rho = \frac{r}{a_B}$ is the dimensionless radial coordinate, $a_B = \frac{\hbar}{mc\alpha}$ is the Bohr radius and $\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c}$ is the fine-structure constant. The formula for the discrete energy levels is

$$E_N = -\frac{1}{2}mc^2\alpha^2 \frac{1}{N^2}, \quad N = l+1, l+2, \dots; \quad (5)$$

for the complete solution of (4), see [2].

In the case of the 3D isotropic harmonic oscillator with potential V_H , the equation of the radial part is

$$R''(\xi) + \frac{2}{\xi}R'(\xi) + \left(2\varepsilon - \xi^2 - \frac{l(l+1)}{\xi^2} \right) R(\xi) = 0, \quad (6)$$

where $\varepsilon = \frac{E}{\hbar\omega}$ is the dimensionless energy and $\xi = \frac{r}{\sqrt{\frac{\hbar}{m\omega}}}$ is the dimensionless radial coordinate, for the complete solution of (6), see [3].

2.2 Radial WKB approximation and its demonstration on the 3D isotropic harmonic oscillator

The WKB method is a technique for obtaining approximate solution to the time-independent Schrödinger equation, see [4, 5].

In the '1D' WKB approximation, the wave function $\psi(x)$ is defined for $x \in \mathbb{R}$, whereas for rotationally invariant potentials the radial wave function $R(r)$ is defined for $r \in \mathbb{R}_0^+$; hence the 'standard' WKB method cannot be used directly. To avoid this problem, region $0 \leq r < \infty$ can be mapped to $-\infty < x < \infty$ by the bijective transformation² $r = e^x$. The radial

wave function is being searched in a form

$$R(r = e^x) \rightarrow R(x) = U(x)e^{f(x)}, \quad (7)$$

where the function $f(x)$ is chosen such that the first derivative of $U(x)$ disappears from the radial Schrödinger equation (3). In the present case, $f(x) = -x/2$, thus we get

$$0 = U''(x) + \underbrace{\frac{e^{2x}}{\hbar^2} 2m \left(E - V(e^x) - \frac{(l+\frac{1}{2})^2 \hbar^2}{2me^{2x}} \right)}_{p^2(e^x)} U(x). \quad (8)$$

Hence, the momentum $p(r)$ of the particle can be picked out:

$$p(r) = \sqrt{2m \left(E - V(r) - \frac{(l+\frac{1}{2})^2 \hbar^2}{2mr^2} \right)}, \quad (9)$$

to be substituted to the WKB condition

$$\frac{1}{\hbar} \int_{r_1}^{r_2} dr p(r) = \left(n + \frac{1}{2} \right) \pi, \quad n \in \mathbb{Z}_0^+, \quad (10)$$

where the limits of the integration $\{r_1, r_2; 0 < r_1 < r_2\}$ are defined by relation $p(r_1) = p(r_2) = 0$. Equation (10) is a condition on energy levels E .

We can check the validity and the precision of this method on the potential V_H . Using equations (9) and (10), we get

$$\left(n + \frac{1}{2} \right) \pi = \frac{1}{2} \int_{t_1}^{t_2} \frac{dt}{t} \sqrt{-t^2 + 2\varepsilon t - \left(l + \frac{1}{2} \right)^2}, \quad (11)$$

where we used the substitution $\xi^2 = t$. For further calculations we take advantage of the following integral:

$$\int_{c_1}^{c_2} \frac{dx}{x} \sqrt{(x-c_1)(c_2-x)} = \frac{\pi}{2} (\sqrt{c_2} - \sqrt{c_1})^2, \quad (12)$$

see [5]. Thus, we get

$$\left(n + \frac{1}{2} \right) \pi = \frac{\pi}{2} \left[\varepsilon - \left(l + \frac{1}{2} \right)^2 \right]. \quad (13)$$

From the previous equation we regain the accurate energy spectrum for 3D isotropic harmonic oscillator:

$$(E_{nl})_{\text{WKB}} = \hbar\omega \left(2n + l + \frac{3}{2} \right) = E_{nl}, \quad (14)$$

¹In particular cases, it will be more convenient to tackle the radial wave function $R_l(r)$ in a form $R_l(r) = K_l(r)r^l$ where the term r^l describes the behaviour of $R_l(r \rightarrow 0)$.

²Since x is an auxiliary variable, we will not pay attention to its physical dimension.

see [3]. Quantum number n is independent from the orbital quantum number l ; the obtained formula leads to the same degeneracy of energy levels and the same parity of states as we would have recovered if separation of variables in cartesian coordinate system had been utilized. However, we must admit that we recovered the exact energy spectrum by the merest chance. It cannot be forgotten that the WKB method works the best for large momentum, small de Broglie wavelength ($d\lambda_{\text{de Broglie}}/dx \ll 2\pi$) and hence for large quantum numbers: n and l (in the present case).

2.3 Quantum mechanics in noncommutative space

In the next two subsections, we will focus on the relations which describe the mathematics in NCQM. For more details we refer the interested readers to original paper [1].

In standard QM, physical quantities are represented by hermitian operators acting on the states of the Hilbert space $\mathcal{H}$ – on the wave functions. In QM with NC coordinates, this idea is still preserved, but the NC wave functions are composed of operators acting in the auxiliary Fock space. The NC space $\mathbb{R}_\lambda^3$ can be described by a model of concentric fuzzy spheres with increasing radius. The commutator of coordinates is defined as

$$[x_i, x_j] = 2i\lambda \varepsilon_{ijk} x_k, \quad (15)$$

where the parameter λ describes the fuzziness of the space structure³. The Fock space is accompanied by two sets of creation $a^\dagger$ and annihilation a operators; their commutation relations look like

$$[a_\alpha, a_\beta^\dagger] = \delta_{\alpha\beta}, \quad [a_\alpha, a_\beta] = [a_\alpha^\dagger, a_\beta^\dagger] = 0, \quad (16)$$

with $\alpha, \beta \in \{1, 2\}$.

The appealing fact is that all the operators of all the relevant physical quantities can be constructed with this choice of the operators $a^\dagger$ and a .

The definition of position operator⁴

$$x_j = \lambda \sigma_{\alpha\beta}^j a_\alpha^\dagger a_\beta, \quad j \in \{1, 2, 3\} \quad (17)$$

³Note that we would have a similar looking commutation relation for a single fuzzy sphere. The difference is that in that case, operator (18) would be restricted to have a fixed value, restricting us onto a subset of the Fock space. Here, we allow any value of r , leading to spheres of all radii.

⁴The operators acting on the states of the Fock space are marked without hat.

obeys commutator (15), σ^j is the corresponding Pauli matrix.

The magnitude of the position vector is defined as

$$r = \lambda (a_\alpha^\dagger a_\alpha + 1); \quad (18)$$

for further calculations we take the advantage of the operator

$$\tilde{r} = \lambda (a_\alpha^\dagger a_\alpha). \quad (19)$$

The wave functions $\Psi(x_j)$ can be expressed instead of x_j in terms of $a^\dagger$ and a in the way:

$$\Psi = \sum C_{m_1 m_2 n_1 n_2} (a_1^\dagger)^{m_1} (a_2^\dagger)^{m_2} (a_1)^{n_1} (a_2)^{n_2}. \quad (20)$$

The number of $a^\dagger$ and a are equal in the definition of x_j , thus they must be equal in the expression for Ψ , too, so $m_1 + m_2 = n_1 + n_2$, where $\{m_1, m_2, n_1, n_2\} \in \mathbb{Z}_0^+$. Moreover, for the arrangement of operators $a^\dagger$ and a , the normal ordering is imposed by hand.

The NC analogy of the method of separation the variables in standard QM (2) leads to the separated form

$$\Psi_{lm} = \lambda^l \sum_{(lm)} \frac{(a_1^\dagger)^{m_1} (a_2^\dagger)^{m_2}}{m_1! m_2!} : \mathcal{K}_l(\tilde{r}) : \frac{a_1^{n_1} (-a_2)^{n_2}}{n_1! n_2!}, \quad (21)$$

where $\mathcal{K}_l(\tilde{r})$ is the NC analogy of the radial wave function $K_l(r)$, related to $R_l(r)$ by $R_l(r) = K_l(r)r^l$, constructed in the same way as (20), and between the colon marks normal ordering is needed. The summation goes over all quantum numbers $l = m_1 + m_2 = n_1 + n_2$ and $m = 1/2 (m_1 - m_2 - n_1 + n_2)$.

The action of the operator $\hat{r}$ from (18) on the wave function⁵ in (21) can be calculated:

$$\hat{r} \Psi_{lm} = \lambda^l \sum_{(lm)} \frac{(a_1^\dagger)^{m_1} (a_2^\dagger)^{m_2}}{m_1! m_2!} : [(\tilde{r} + \lambda l + \lambda) \mathcal{K} + \lambda \tilde{r} \mathcal{K}'] : \frac{a_1^{n_1} (-a_2)^{n_2}}{n_1! n_2!}, \quad (22)$$

where $\mathcal{K} \equiv \mathcal{K}_l(\tilde{r})$ and similarly for derivatives.

The definition of the Laplace operator in $\mathcal{H}_\lambda$ is⁶

$$\hat{\Delta}_\lambda \Psi = -\frac{1}{\lambda r} [\hat{a}_\alpha^\dagger, [\hat{a}_\alpha, \Psi]]; \quad (23)$$

⁵The operators acting on the states of the Hilbert space $\mathcal{H}_\lambda$ are marked with hat.

⁶Note that in this case the creation and annihilation operators are rewarded with hat since in the presence case they act on the wave function. We want to stress that marking the operator with hat is independent from the operator itself, the decisive factor is whether the operator is understood as acting on the states of the Fock space or the Hilbert space $\mathcal{H}_\lambda$.

the action of the operator $[\hat{a}_\alpha^\dagger, [\hat{a}_\alpha, \cdot]]$ on the wave function in (21) is

$$[\hat{a}_\alpha^\dagger, [\hat{a}_\alpha, \Psi_{lm}]] = \lambda^l \sum_{(lm)} \frac{(a_1^\dagger)^{m_1} (a_2^\dagger)^{m_2}}{m_1! m_2!} : [-\lambda \tilde{r} \mathcal{K}'' - 2(l+1)\lambda \mathcal{K}'] : \frac{a_1^{n_1} (-a_2)^{n_2}}{n_1! n_2!} . \quad (24)$$

Putting together our knowledge of the NC potential and the NC Laplace operator, we acquire the Hamiltonian operator $\hat{H}_\lambda$ in the form

$$\hat{H}_\lambda \Psi \equiv \left[-\frac{\hbar^2}{2m} \hat{\Delta}_\lambda + V(\hat{r}) \right] \Psi . \quad (25)$$

2.4 Exact solution of the NC Coulomb potential

Based on (25), the NC analog of the Schrödinger equation with the Coulomb potential V_C in $\mathbb{R}_\lambda^3$ is

$$\frac{\hbar^2}{2m\lambda r} [\hat{a}_\alpha^\dagger, [\hat{a}_\alpha, \Psi]] - \frac{1}{4\pi\epsilon_0} \frac{e^2}{r} \Psi = E \Psi . \quad (26)$$

Multiplying⁷ the previous equation with $\hat{r}$ and inserting (24) and (22), we get

$$: \tilde{\rho} \mathcal{K}'' + [-\epsilon \sigma \tilde{\rho} + 2(l+1)] \mathcal{K}' + [-\epsilon \tilde{\rho} - \epsilon \sigma(l+1) + 2] \mathcal{K} : = 0 , \quad (27)$$

where $\tilde{\rho}$ is the dimensionless operator $\tilde{r}$ from (19), and this equation is the radial Schrödinger equation in NCQM expressed with $\mathcal{K}$. Equation (27) can be interpreted in the way that the configuration of the operators on the left side results in a zero operator. In the case we remove the colon marks, we get the differential equation

$$\rho K'' + [-\epsilon \sigma \rho + 2(l+1)] K' + [-\epsilon \rho - \epsilon \sigma(l+1) + 2] K = 0 , \quad (28)$$

for an ordinary function K of an ordinary variable ρ . The claim is that one obtains solution to (27) by replacing ρ in the solution to (28) with operator $\tilde{\rho}$, see [1].

The solution of (28) for the radial wave function $K(\rho)$ results in hypergeometric function, and the exact energy spectrum of the bound states is

$$E_N^\lambda = \frac{\hbar^2}{m\lambda^2} \left(1 - \sqrt{1 + \frac{m^2 c^2 \alpha^2 \lambda^2}{\hbar^2 N^2}} \right) . \quad (29)$$

⁷Expression $\frac{1}{\hat{r}}$ can be interpreted as the inverse operator to the operator $\hat{r}$.

One can see that the energy levels calculated in standard QM increase due to the effect of the NC space structure. Taking the limit $\lambda \rightarrow 0$ we recover the energy spectrum (5) in QM.

3 Solution of NC Coulomb potential by WKB approximation

Utilizing relation $K\rho^l = R$ and introducing $\sigma = \lambda/a_B$, equation (28) is obtained in the form

$$R''(\rho) + \left(\frac{2}{\rho} + \sigma(-\epsilon) \right) R'(\rho) + \left(-\epsilon + \frac{2}{\rho} - \frac{l(l+1)}{\rho^2} + \sigma \frac{-\epsilon}{\rho} \right) R(\rho) = 0 , \quad (30)$$

which is the analog of the radial Schrödinger equation in NCQM. Substitution $\rho = e^x$ is used again to rewrite the dimensionless radial equation and provided that we consider the radial wave function in the form of (7) where $f(x) = \frac{1}{2}(\sigma \epsilon e^x - x)$, the previous equation simplifies to

$$0 = U''(x) + \underbrace{e^{2x} \left(-\epsilon + \frac{2}{e^x} - \frac{(l+\frac{1}{2})^2}{e^{2x}} - \sigma^2 \frac{\epsilon^2}{4} \right)}_{\kappa^2(e^x)} U(x) , \quad (31)$$

and $U'(x)$ term disappears. From this equation one can find the dimensionless momentum of the electron κ , which goes to the WKB condition (10), so we get

$$\left(n + \frac{1}{2} \right) \pi = \int_{\rho_1}^{\rho_2} d\rho \underbrace{\sqrt{-\epsilon + \frac{2}{\rho} - \frac{(l+\frac{1}{2})^2}{\rho^2} - \sigma^2 \frac{\epsilon^2}{4}}}_{\kappa(\rho)} . \quad (32)$$

Using a little algebra, the previous equation becomes

$$\left(n + \frac{1}{2} \right) \pi = \int_{\rho_1}^{\rho_2} \frac{d\rho}{\rho} \sqrt{\epsilon \left(1 + \frac{\sigma^2}{4} \epsilon \right)} \times \sqrt{-\rho^2 + \frac{2}{\epsilon \left(1 + \frac{\sigma^2}{4} \epsilon \right)} \rho - \frac{(l+\frac{1}{2})^2}{\epsilon \left(1 + \frac{\sigma^2}{4} \epsilon \right)}} , \quad (33)$$

where (12) can be utilized. Thus,

$$\pi \left[\frac{1}{\sqrt{\epsilon \left(1 + \frac{\sigma^2}{4} \epsilon \right)}} - \left(l + \frac{1}{2} \right) \right] = \left(n + \frac{1}{2} \right) \pi . \quad (34)$$

We can express $\varepsilon > 0$ from the previous equation and by introducing a new quantum number $N \equiv (n + l + 1) \in \mathbb{N}$ we shall write the final formula

$$(\varepsilon_N^\sigma)_{\text{WKB}} = -\frac{2}{\sigma^2} \left(1 - \sqrt{1 + \frac{\sigma^2}{N^2}} \right). \quad (35)$$

Switching back to quantities with dimension, we get

$$(E_N^\lambda)_{\text{WKB}} = \frac{\hbar^2}{m\lambda^2} \left(1 - \sqrt{1 + \frac{m^2 c^2 \alpha^2 \lambda^2}{\hbar^2 N^2}} \right). \quad (36)$$

By WKB approximation we recovered the same modified energy spectrum as (29).

$$\begin{aligned} \hat{r}^3 \Psi_{lm} \equiv \hat{r} (\hat{r} (\hat{r} \Psi_{lm})) = \lambda^l \sum_{(lm)} \frac{(a_1^\dagger)^{m_1} (a_2^\dagger)^{m_2}}{m_1! m_2!} : \left[\tilde{r}^3 \mathcal{K} + \lambda \left(3(l+2) \tilde{r}^2 \mathcal{K} + 3 \tilde{r}^3 \mathcal{K}' \right) + \right. \\ \left. + \lambda^2 \left((3l^2 + 9l + 7) \tilde{r} \mathcal{K} + 6(l+2) \tilde{r}^2 \mathcal{K}' + 3 \tilde{r}^3 \mathcal{K}'' \right) + \right. \\ \left. + \lambda^3 \left((l+1)^3 \mathcal{K} + (3l^2 + 9l + 7) \tilde{r} \mathcal{K}' + 3(l+2) \tilde{r}^2 \mathcal{K}'' + \tilde{r}^3 \mathcal{K}''' \right) \right] : \frac{a_1^{n_1} (-a_2)^{n_2}}{n_1! n_2!}. \quad (38) \end{aligned}$$

By the introduction of $\sigma = \lambda / \sqrt{\frac{\hbar}{m\omega}}$ – the dimensionless parameter, equation (37) thus

4 Solution of 3D isotropic harmonic oscillator by WKB approximation

Based on (25), the NC analog of the Schrödinger equation with isotropic harmonic oscillator potential V_H in $\mathbb{R}_\lambda^3$ is

$$\frac{\hbar^2}{2m\lambda r} [\hat{a}_\alpha^\dagger, [\hat{a}_\alpha, \Psi]] + \frac{1}{2} m \omega^2 \hat{r}^2 \Psi = E \Psi. \quad (37)$$

It is inevitable to calculate the action of operator $\hat{r}^3$ on the wave function Ψ_{lm} , with the help of (22), we get

yields the radial Schrödinger equation in the NC configuration space in the following form:

$$\begin{aligned} 0 = R'' + \frac{2}{\xi} R' - \frac{l(l+1)}{\xi^2} R - \xi^2 R + 2\varepsilon R + \sigma \left(2\varepsilon R' + \frac{2\varepsilon R}{\xi} - 3\xi^2 R' - 6\xi R \right) + \\ + \sigma^2 \left(-3\xi^2 R'' - 12\xi R' - 7R \right) + \sigma^3 \left(-\xi^2 R''' - 6\xi R'' - 7R' - \frac{R}{\xi} \right). \quad (39) \end{aligned}$$

Analogously to the described method in subsection 2.2, we substitute $\xi = e^x$ and find the proper $f(x)$

in (7) such that $U'(x)$ term will not be present. By neglecting all the higher terms in the parameter σ , we acquire the previous equation in the form

$$0 = U''(x) + e^{2x} \underbrace{\frac{2 \left(\varepsilon - \frac{e^{2x}}{2} - \frac{(l+\frac{1}{2})^2}{2e^{2x}} + \frac{\sigma^2}{2} \left[\frac{27}{4} e^{4x} - \varepsilon^2 - 15\varepsilon e^{2x} + 5 + 9l(l+1) \right] \right)}{1 - 12\sigma^2 e^{2x}}} U(x). \quad (40)$$

$\kappa^2(e^x)$

One can see that $\kappa^2(e^x)$ does not depend on σ^1 , the first corrections emerge with the terms directly proportional to σ^2 . This indicates that the first correction in energy spectrum comes by the factor σ^2 . By neglecting all the terms in the orders 3, 4, ... of the parameter σ , formula for $\kappa^2(\xi)$ can be written in the

following form:

$$\begin{aligned} \kappa^2(\xi) = 2 \left(\varepsilon - \frac{\xi^2}{2} - \frac{(l+\frac{1}{2})^2}{2\xi^2} + \frac{\sigma^2}{8} [8 - \right. \\ \left. - 4\varepsilon^2 - 12l(l+1) + 36\varepsilon\xi^2 - 21\xi^4] \right). \quad (41) \end{aligned}$$

Hence, the WKB condition is attained:

$$\left(n + \frac{1}{2} \right) \pi = \int_{\xi_1}^{\xi_2} d\xi \sqrt{2\varepsilon - \xi^2 - \frac{(l+\frac{1}{2})^2}{\xi^2} + \frac{\sigma^2}{4} [8 - 4\varepsilon^2 - 12l(l+1) + 36\varepsilon\xi^2 - 21\xi^4]}. \quad (42)$$

We take the advantage of further simplification of substitution $\xi^2 = t$ to get the previous equation in

a more straightforward form to calculate the integral:

$$\begin{aligned} \left(n + \frac{1}{2} \right) \pi = \frac{1}{2} \left(\frac{\sigma}{2} \sqrt{21} \right) \int_{t_1}^{t_2} \frac{dt}{t} \times \\ \times \sqrt{-\frac{4}{21\sigma^2} \left(l + \frac{1}{2} \right)^2 + \left(\frac{8}{21\sigma^2} \varepsilon + \frac{8}{21} - \frac{4}{21} \varepsilon^2 - \frac{12}{21} l(l+1) \right) t + \left(\frac{36}{21} \varepsilon - \frac{4}{21\sigma^2} \right) t^2 - t^3}. \quad (43) \end{aligned}$$

One can see that there is a cubic algebraic expression lurking under the square root. Calculating the roots of equation $\kappa(t) = 0$, we attain three solutions $\{t_1, t_2, t_0; t_0 < 0 < t_1 < t_2\}$:

$$t_0 = -\frac{4}{21\sigma^2} - \frac{2\varepsilon}{7} + \sigma^2 \left(4\varepsilon^2 - \frac{9l(l+1)}{4} - \frac{53}{16} \right) + \dots,$$

$$t_1 = \left(\varepsilon - \frac{1}{2} \sqrt{4\varepsilon^2 - (2l+1)^2} \right) + \sigma^2 \left(\frac{\varepsilon [64\varepsilon^2 - 60l(l+1) - 59]}{16\sqrt{4\varepsilon^2 - (2l+1)^2}} - 2\varepsilon^2 + \frac{1}{32} [36l(l+1) + 53] \right) + \dots,$$

$$t_2 = \left(\varepsilon + \frac{1}{2} \sqrt{4\varepsilon^2 - (2l+1)^2} \right) + \sigma^2 \left(\frac{\varepsilon [-64\varepsilon^2 + 60l(l+1) + 59]}{16\sqrt{4\varepsilon^2 - (2l+1)^2}} - 2\varepsilon^2 + \frac{1}{32} [36l(l+1) + 53] \right) + \dots$$

The limits of integration are defined by $\{t_1, t_2\}$. Taking the limit $\sigma \rightarrow 0$, for $\{t_1, t_2\}$ we recover the roots of the quadratic algebraic expression under the square root in (11), and the third root disappears from the standard QM because t_0 approaches to $-\infty$.

The integral in (43) can be rewritten by introducing new variables $q = t/t_1$, $q_0 = t_0/t_1$ and $q_2 = t_2/t_1$. To calculate this integral, software Mathematica gave us a hand:

$$\begin{aligned} t_1^{3/2} \int_1^{q_2} \frac{dq}{q} \sqrt{(q-q_0)(q-1)(q_2-q)} = \\ = t_1^{3/2} \frac{2}{3\sqrt{q_2-q_0}} \left[3q_0 \Pi \left(\frac{q_2-1}{q_2} \middle| \frac{q_2-1}{q_2-q_0} \right) - \right. \\ \left. - [q_0(-q_0+q_2+1) + 2q_2] K \left(\frac{q_2-1}{q_2-q_0} \right) - \right. \\ \left. - (q_0-q_2)(q_0+q_2+1) E \left(\frac{q_2-1}{q_2-q_0} \right) \right], \quad (44) \end{aligned}$$

where E is the complete elliptic integral, K is the complete elliptic integral of the first kind and Π is the complete elliptic integral of the third kind, see [6]. The validity of this result was verified by numerical integration, too.

Applying this integral to (43), rewriting the original variables and neglecting all the terms in orders 3, 4, ... of the parameter σ , we proceed to the equation

$$\left(n + \frac{1}{2} \right) \pi = \frac{\pi}{2} \left[\varepsilon - \left(l + \frac{1}{2} \right) \right] + \sigma^2 \frac{\pi}{128} [4\varepsilon^2 - 12l(l+1) + 85]. \quad (45)$$

Generally, for further calculations an expansion of ε in the small perturbation of σ would have been presented. However, (45) is just a quadratic equation for ε , we simply write down the only positive root up to the terms in the 2nd order of σ :

$$\begin{aligned} (\varepsilon_{nl}^\sigma)_{\text{WKB}} = \left(2n + l + \frac{3}{2} \right) + \\ + \frac{\sigma^2}{32} [4(l^2 - 2ln - 3n - 2n^2) - 47] + \dots \end{aligned} \quad (46)$$

In the acquired formula, we demand from the correction in energy to be on a perturbative scale, thus $n, l \ll \sigma^{-1}$. On the other hand, we once again emphasize that WKB approximation works the best for large energy-defining quantum numbers, thus $n, l \gg 1$. Since $\sigma \approx 10^{-25}$, our two conditions do not contradict⁸.

We vow that the modified energy spectrum up to the first non-zero modification due to the NC structure of space looks like

$$\begin{aligned} (E_{nl}^\lambda)_{\text{WKB}} = \hbar\omega \left(2n + l + \frac{3}{2} \right) + \\ + \lambda^2 \frac{m\omega^2}{32} [4(l^2 - 2ln - 3n - 2n^2) - 47] + \dots \end{aligned} \quad (47)$$

The first remark to the acquired formula is that the energy is now dependent on the mass of the particle m , which was expected due to dimensional reasons. Next, the energy levels do not shift only in one direction: e.g. energy level defined

⁸We need order estimation of the typical distance $\sqrt{\frac{\hbar}{m\omega}}$. Between potential applications of this theoretical prediction we will hint at the Lennard-Jones intermolecular potential $V_{LJ}(r) = 4V_0 \left[\left(\frac{s}{r} \right)^{12} - \left(\frac{s}{r} \right)^6 \right]$, where $V_0 \approx \text{meV}$ is the dispersion energy and $s \approx \text{\AA}$ is the size of the atom. The potential can be expanded around the equilibrium distance r_0 , which is defined by relation $V'_{LJ}(r_0) = 0$, to get $V''_{LJ}(r_0) = m\omega^2$, which results for $m \approx 10^{-27} \text{ kg}$ estimation $\omega \approx 10^{13} \text{ s}^{-1}$, hence $\sqrt{\frac{\hbar}{m\omega}} \approx 10^{-10} \text{ m}$.

by $n = 0, l = 0$ decreases, i.e. $(E_{00}^\lambda)_{\text{WKB}} < E_{00}$, whereas energy level defined by $n = 0, l = 4$ increases, i.e. $(E_{04}^\lambda)_{\text{WKB}} > E_{04}$. Eventually, taking the limit $\lambda \rightarrow 0$ we recover the energy spectrum (14) in QM. Although we used an approximation method, for standard QM the derived energy spectrum (14) happened to be correct. Hence, we can be optimistic that the modified energy formula (47) is correct for the whole spectrum, too.

5 Conclusion

To conclude, let us summarize our results. We have analyzed by radial WKB method the modification of energy levels in NCQM to the standard QM spectra of the bound states in Coulomb potential and 3D isotropic harmonic oscillator. Despite the WKB method being just an approximation, we obtained the exact same modified energy levels for the Coulomb potential as in [1]. Being elated by the precision of the WKB approximation, we have acquired the first non-zero correction in the energy spectrum of the 3D isotropic harmonic oscillator, which comes with the modification in order 2 of the parameter of the space-noncommutativity λ .

The NC predictions do not refute the experimentally verified QM results, since taking the limit $\lambda \rightarrow 0$ both QM and NCQM coincide. However, the potential noncommutativity of the configuration space of the 3D isotropic harmonic oscillator would appear in the spectrum of the molecular vibration, molar heat capacity of diatomic gas and the heat capacity of solid materials. Obviously, today the effect of noncommutativity in the experimental data would be beyond the effect of other more significant perturbations. However, provided that we will succeed in excluding all the disturbances in the measurement and carry it out with adequate sensitive gauges, the noncommutativity could be in principle experimentally exposed.

It remains an open problem to calculate the 3rd order terms of λ in the energy spectrum of 3D isotropic harmonic oscillator, since during the manipulations with (39) all the terms in order 3, 4, ... of λ got neglected. In addition, 3D harmonic oscillator can be described in an algebraic way through $SU(3)$ group and when considering NC space structure this approach could happen to be more amenable. Moreover, to find further measurable effects of the NC space structure, it would be pleasing to apply our model to the Cornwell potential, which describes the

potential between two quarks, where the effect of NC space structure would be in principle observable in the masses of the quarkonium states.

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