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Second order kinetic term effective actions for matrix model description of fuzzy field theories

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Abstract: We study scalar field theory defined on noncommutative spaces. We derive the kinetic term effective action up to the second order for arbitrary fuzzy space. We utilize the acquired result to reproduce the previously obtained kinetic term effective action for the fuzzy sphere and to derive the kinetic term effective action for Grosse-Wulkenhaar matrix model defined on noncommutative plane by truncation of the Heisenberg algebra.

Keywords: scalar field theory, kinetic term effective action, fuzzy sphere, Grosse-Wulkenhaar matrix model

1 Introduction

The noncommutative quantum field theory is the application of the noncommutative mathematics to the spacetime of the quantum field theory with such a modification that the coordinates become noncommutative. As a consequence, noncommutativity blurs the space, the precise coordinates of a single point lose their meaning, the spacetime becomes fuzzy.

In the present work, we are interested in the real scalar quantum field theory defined on the fuzzy sphere S_F^2 and the the Grosse-Wulkenhaar matrix model (GW model) on noncommutative plane.

The quantum field theory can be defined via the path integral formulation, where the functional integral is ill-defined. However, the fuzzy models can be described as finite dimensional matrices, thus we remedy that extant problem. The integration becomes the trace over the polynomial of the matrix, which could simplify the field theory to eigenvalue formulation. However, the presence of the kinetic term reshuffles these attempts.

Our method to overcome this trouble is to present an expansion of the kinetic term, and we handle the

resulting expressions using group theory methods. These methods were developed in [1, 2].

In this paper, to the best of our knowledge, we are the first who derive the second order term in the expansion of an arbitrary kinetic term. We use the derived relations to affirm the validity of the kinetic term effective action up to the second order for the fuzzy sphere [2] (where the specific kinetic term enabled the calculations without tackling the kinetic term generally), and as a new result, we derive the kinetic term effective action up to the second order for the GW model.

The paper is structured as follows. In section 2, we render some common information about the fuzzy sphere, the truncated Heisenberg algebra, we briefly describe the noncommutative quantum field theory and within that the GW model. In section 3, we demonstrate the integration over the angular part containing the kinetic term and present our Lie algebra conventions and relations. In section 4, we calculate the perturbative expansion in the first and the second order. We utilize the obtained results for the fuzzy sphere and the GW model in section 5.

2 Description of our models

2.1 Fuzzy sphere

The most understood fuzzy space is the fuzzy sphere S_F^2 . Let us trigger its brief description from the regular sphere S^2 with radius R , which is embedded in the Euclidean manifold $\mathbb{R}^3$ with coordinates x_i in the way that the origin of $\mathbb{R}^3$ coincides with the center of S^2 . Thus, we get

$$\sum_{i=1}^3 x_i^2 = R^2 \quad \text{with} \quad [x_i, x_j] \equiv x_i x_j - x_j x_i = 0, \quad (1)$$

where the coordinates of the regular sphere S^2 commute.

The fuzzy sphere S_F^2 is obtained by generalization of

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the commutator in (1) to the case

$$\sum_{i=1}^3 \tilde{x}_i^2 = \mathcal{R}^2 \quad \text{with} \quad [\tilde{x}_i, \tilde{x}_j] = i\theta \varepsilon_{ijk} \tilde{x}_k, \quad (2)$$

where i is the imaginary unit and we will use the Einstein summation convention. In relation (2) the coordinates $\tilde{x}_i$ of $\mathbb{R}^3$ no longer commute, the parameter θ describes the scale of fuzzyness and $\mathcal{R}$ can be reasonably interpreted as the radius of the fuzzy sphere.

To obtain the commutator in (2), we recall the defining representation of $\mathfrak{su}(2)$, where L_i 's are the generators of the correspondent $\mathfrak{su}(2)$ Lie algebra, for which the following identities hold in the spin- j representation:

$$[L_i, L_j] = i\varepsilon_{ijk} L_k \quad \text{and} \quad \sum_{i=1}^3 L_i^2 = j(j+1)\mathbb{1}, \quad (3)$$

where $\mathbb{1}$ denotes the $N \times N$ identity matrix. Via the choice

$$\tilde{x}_i = \frac{2\mathcal{R}}{\sqrt{N^2-1}} L_i \quad \text{and} \quad \theta = \frac{2\mathcal{R}}{\sqrt{N^2-1}}, \quad (4)$$

one recovers (3), where $N = 2j+1$ is the dimension of the irreducible representation. Note two remarks: the coordinates $\tilde{x}_i$ of the fuzzy sphere have become matrices, furthermore, taking the limit $N \rightarrow \infty$, the noncommutative parameter $\theta \rightarrow 0$, hence we recover the regular sphere S^2 .

2.2 Truncated Heisenberg algebra

An other frequently used noncommutative space is the Heisenberg algebra, where the space is described by non-commuting position operators $\hat{x}$ and $\hat{y}$, their commutator – the space noncommutativity is constant, i.e.

$$[\hat{x}, \hat{y}] = i, \quad (5)$$

where this commutator resembles the Heisenberg's uncertainty principle from quantum mechanics.

This Heisenberg algebra (5) can be represented in the energy representation of the linear harmonic oscilla-

tor by infinite dimensional matrices as

$$\hat{x} = \frac{1}{\sqrt{2}} \begin{pmatrix} & +\sqrt{1} & & & \\ +\sqrt{1} & & +\sqrt{2} & & \\ & +\sqrt{2} & & \ddots & \\ & & \ddots & & +\sqrt{N-1} \\ & & & +\sqrt{N-1} & \ddots \end{pmatrix},$$

$$\hat{y} = \frac{i}{\sqrt{2}} \begin{pmatrix} & -\sqrt{1} & & & \\ +\sqrt{1} & & -\sqrt{2} & & \\ & +\sqrt{2} & & \ddots & \\ & & \ddots & & -\sqrt{N-1} \\ & & & +\sqrt{N-1} & \ddots \end{pmatrix}. \quad (6)$$

Truncating these matrices to $N \times N$ finite dimensional matrices, we get

$$X = \frac{1}{\sqrt{2}} \begin{pmatrix} & +\sqrt{1} & & & \\ +\sqrt{1} & & +\sqrt{2} & & \\ & +\sqrt{2} & & \ddots & \\ & & \ddots & & +\sqrt{N-1} \\ & & & +\sqrt{N-1} & \end{pmatrix},$$

$$Y = \frac{i}{\sqrt{2}} \begin{pmatrix} & -\sqrt{1} & & & \\ +\sqrt{1} & & -\sqrt{2} & & \\ & +\sqrt{2} & & \ddots & \\ & & \ddots & & -\sqrt{N-1} \\ & & & +\sqrt{N-1} & \end{pmatrix}, \quad (7)$$

thus the Heisenberg algebra (5) alters to the so-called truncated Heisenberg algebra

$$[X, Y] = i(\mathbb{1} - NP_N), \quad (8)$$

where P_N is the projector

$$P_N = \begin{pmatrix} \mathbb{0} & \\ & 1 \end{pmatrix}, \quad (9)$$

here $\mathbb{0}$ is the $(N-1) \times (N-1)$ zero matrix. Matrices X and Y describe a finite part of the two-dimensional plane, taking limit $N \rightarrow \infty$, we recover the complete non-commutative $\hat{x}$ - $\hat{y}$ plane. In this limit, the truncated Heisenberg algebra (8) develops back to the Heisenberg algebra (5), and thus (7) can be considered as matrix versions of coordinate operators on the non-commutative plane.

2.3 Scalar field theory on the models

The subject of our interest is the scalar field theory defined on these models. The scalar field theory can

be formulated through the path integrals, where the mean value of a function f is defined as

$$\langle f \rangle = \frac{1}{Z} \int [d\Phi] f(\Phi) e^{-\int \Omega(\frac{1}{2}\Phi\Delta\Phi + g_2\Phi^2 + g_4\Phi^4)} \quad \text{with} \\ Z = \int [d\Phi] e^{-\int \Omega(\frac{1}{2}\Phi\Delta\Phi + g_2\Phi^2 + g_4\Phi^4)}, \quad (10)$$

where the integrals are taken over all field configurations, Z is the generating functional and Ω is the volume form of the space we are interested in. The conceptual problem with this relation arises when one attempts to integrate over all the possible field configurations. Indeed, this integration is not well defined. Fortunately, this concern can be removed when we redefine the quantum field theory in matrix formalism.

Considering real scalar theory, the field becomes hermitian $N \times N$ matrix Φ . The integration over the Lagrangian becomes the trace of the matrices. We get

$$\langle f \rangle = \frac{1}{Z} \int [d\Phi] f(\Phi) e^{-\text{tr}(\Phi\mathcal{K}\Phi + g_2\Phi^2 + g_4\Phi^4)} \quad \text{with} \\ Z = \int [d\Phi] e^{-\text{tr}(\Phi\mathcal{K}\Phi + g_2\Phi^2 + g_4\Phi^4)}, \quad (11)$$

where the Laplace operator in the kinetic term has transformed to the operator $\mathcal{K}$. The integration over all the possible paths characterizing how the field Φ evolves is suddenly defined comprehensible, one needs to integrate over all the matrix configurations $\Phi(N)$. Since Φ is hermitian, we can diagonalize it: $\Phi = U\Lambda U^\dagger$, where U is an unitary matrix and $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_i, \dots, \lambda_N)$ is the diagonal matrix of eigenvalues of Φ . In the case of the function that does not depend on the angular degrees of freedom U , i.e. $f(\Phi) = f(\Lambda) \neq f(\Lambda, U)$, the mean value in (11) can be rewritten to¹

$$\langle f \rangle = \frac{1}{Z} \int [d\Lambda] f(\Lambda) e^{-\text{tr}(g_2\Lambda^2 + g_4\Lambda^4) + 2\sum_{i < j} \ln|\lambda_i - \lambda_j|} \times \\ \times \int [dU] e^{-\text{tr}(\Phi\mathcal{K}\Phi)}, \quad (12)$$

thus, the integral in (11) splits up to the product of the integral over the diagonal matrix Λ and the angular part. It is meaningful to tackle such a base T_μ for the matrices $\Phi = \Phi_\mu T_\mu$ which are eigenmatrices of the operator $\mathcal{K}$, so $\mathcal{K}T_\mu = k_\mu T_\mu$, where k_μ 's are the eigenvalues.

¹We assume that the parameters in the exponent are rescaled in the way that the exponent is directly proportional to N^2 in the limit $N \rightarrow \infty$.

In the case of the fuzzy sphere, the definition for the kinetic term is given via the action of operator $\mathcal{K}$ on the field Φ as

$$\mathcal{K}\Phi = \frac{1}{2} [L_i, [L_i, \Phi]], \quad (13)$$

where L_i 's are known from (3). The eigenvalues k_μ are known by the action of $\mathcal{K}$ on its eigenmatrices T_μ as

$$\mathcal{K}T_\mu = k_\mu T_\mu = \frac{1}{2} l(l+1) T_\mu, \quad (14)$$

where $l = 0, 1, \dots, N-1$ determines the current irreducible representation we have been restricted onto from the accessible ones.

Our second model, the Grosse-Wulkenhaar matrix model is a scalar field theory on the truncated Heisenberg algebra defined by the generating functional

$$Z = \int [d\Phi] e^{-\text{tr}(\Phi\mathcal{K}\Phi + g_r R\Phi^2 + g_2\Phi^2 + g_4\Phi^4)}, \quad (15)$$

where $g_r R\Phi^2$ is the harmonic oscillator potential term and the definition for the kinetic term is given via the action of operator $\mathcal{K}$ on the field Φ as

$$\mathcal{K}\Phi = [X, [X, \Phi]] + [Y, [Y, \Phi]], \quad (16)$$

where matrices X and Y were defined in (7). For further information we would like to refer the interested readers to [7]. In this work, we will drop the harmonic oscillator term and concentrate on the kinetic term part.

3 Preliminaries

In this section, we provide some indispensable preliminaries for what follows in further sections. To keep this report as short as possible, we concentrate on the notions and expressions which will be directly used.

3.1 Integrating over angular part via the character expansion

In the context of our work, the angular integral first appeared in [1] where it was tackled as a perturbative expansion in the powers of the kinetic term. Here, we present the brief summary of this method. Let us introduce a small parameter β in which the expansion will be shortly performed. The physical interpretation of β can be the reciprocal of the temperature of

the statistical model, and thus the model is examined for high temperature expansion. We get²

$$I = \int [dU] e^{-\beta \text{tr}(\Phi \mathcal{K} \Phi)} = 1 - \beta \int [dU] \text{tr}(\Phi \mathcal{K} \Phi) + \beta^2 \frac{1}{2} \int [dU] \text{tr}^2(\Phi \mathcal{K} \Phi) + \mathcal{O}(\beta^3). \quad (17)$$

Once the values of the integrals in the expansion

$$I_1 = \int [dU] \text{tr}(\Phi \mathcal{K} \Phi), \quad I_2 = \int [dU] \text{tr}^2(\Phi \mathcal{K} \Phi) \quad (18)$$

are obtained, we can straightforwardly reexponentiate them to get

$$I = e^{-\beta [I_1 + \beta \frac{1}{2} (I_1^2 - I_2)]} + \mathcal{O}(\beta^3) \equiv e^{-\beta [S_1 + \beta S_2]} + \mathcal{O}(\beta^3) \equiv e^{-\beta S_{\text{kin}}^{\text{eff}}} + \mathcal{O}(\beta^3), \quad (19)$$

where S_1 , S_2 and $S_{\text{kin}}^{\text{eff}}$ are defined by this equation.

As we have mentioned, we use that base T_μ for the matrices $\Phi = \Phi_\mu T_\mu$ which are eigenmatrices of the operator $\mathcal{K}$, i.e. $\mathcal{K} T_\mu = k_\mu T_\mu$. In general, the relations for the generators of $\mathfrak{u}(N)$ Lie algebra are valid for the T_μ matrices³.

To integrate out the angular degrees of freedom in Φ , we again decompose the hermitian matrix Φ in the mentioned way $\Phi = U \Lambda U^\dagger$, where U is unitary matrix and $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_N)$. From these we get

$$\Phi_\mu = 2 \text{tr}(U \Lambda U^\dagger T_\mu), \quad (20)$$

where we have used the future identity (39). With the help of this and applying again (39), we get I_1 from (18) in the form

$$\begin{aligned} I_1 &= \frac{1}{2} k_\mu \delta_{\mu\nu} \int [dU] \Phi_\mu \Phi_\nu = \\ &= 2k_\mu \delta_{\mu\nu} \int [dU] \text{tr}(U \Lambda U^\dagger T_\mu) \text{tr}(U \Lambda U^\dagger T_\nu) \equiv \\ &\equiv 2k_\mu \delta_{\mu\nu} I_{\mu\nu}, \end{aligned} \quad (21)$$

where $I_{\mu\nu}$ is defined by this equation.

²We use a simplified notation $\text{tr}^n \Lambda = (\text{tr} \Lambda)^n$.

³We assume that the only eigenmatrix of the kinetic term $\mathcal{K}$ with a nonvanishing trace is the multiple of identity matrix. This is a reasonable assumption, as in the fuzzy field theory, the usual d'Alembert operator $\square$ is replaced by the double commutator $[A_i, [A_i, \cdot]]$, for the set of hermitian matrices A_i that define the particular fuzzy space. For such double-commutator kinetic term, all eigenmatrices but the identity have vanishing trace. However, the formulas presented could be straightforwardly (yet laboriously) generalized for the case where this assumption does not hold anymore.

The key step to solve this kind of integral is to rewrite to the form

$$I_1 = 2k_\mu \delta_{\mu\nu} \int [dU] \text{tr}((U \otimes U)(\Lambda \otimes \Lambda) \times (U^\dagger \otimes U^\dagger)(T_\mu \otimes T_\nu)) \quad (22)$$

According to [1], it is fruitful to use the orthogonality relation for Haar measure

$$\int [dU] \rho_{ij}(U) \rho_{kl}^\dagger(U) = \frac{1}{\dim \rho} \delta_{il} \delta_{jk}, \quad (23)$$

which evaluates the integral

$$\begin{aligned} I_1 &= 2k_\mu \delta_{\mu\nu} I_{\mu\nu} = \\ &= 2k_\mu \delta_{\mu\nu} \sum_\rho \frac{1}{\dim \rho} \text{tr}_\rho(\Lambda \otimes \Lambda) \text{tr}_\rho(T_\mu \otimes T_\nu). \end{aligned} \quad (24)$$

An analogous calculation gives for I_2 from (18):

$$\begin{aligned} I_2 &= \frac{1}{4} k_\mu \delta_{\mu\nu} k_\rho \delta_{\rho\sigma} \int [dU] \Phi_\mu \Phi_\nu \Phi_\rho \Phi_\sigma = \\ &= 4k_\mu \delta_{\mu\nu} k_\rho \delta_{\rho\sigma} \int [dU] \text{tr}(U \Lambda U^\dagger T_\mu) \times \\ &\quad \times \text{tr}(U \Lambda U^\dagger T_\nu) \text{tr}(U \Lambda U^\dagger T_\rho) \text{tr}(U \Lambda U^\dagger T_\sigma) = \\ &= 4k_\mu \delta_{\mu\nu} k_\rho \delta_{\rho\sigma} \sum_\rho \frac{1}{\dim \rho} \text{tr}_\rho(\Lambda \otimes \Lambda \otimes \Lambda \otimes \Lambda) \times \\ &\quad \times \text{tr}_\rho(T_\mu \otimes T_\nu \otimes T_\rho \otimes T_\sigma) \equiv \\ &\equiv 4k_\mu \delta_{\mu\nu} k_\rho \delta_{\rho\sigma} I_{\mu\nu\rho\sigma}, \end{aligned} \quad (25)$$

where $I_{\mu\nu\rho\sigma}$ is defined by this equation.

In the previous two equations, the sum goes over all the irreducible representations the tensor product of the corresponding fundamental representation decomposes into, the trace over the tensor product of Λ matrices is the character of the representation $\chi_\rho(\Lambda)$. Here, we would like to refer to the paper [1], where the relations between the traces in the irreducible representation and the traces in the fundamental representation, the characters and the dimensions of the irreducible representations are presented in a very elegant and well arranged way.

We would like to inform the reader about a typo in [1], the trace of the fourth irreducible representation in the table (B.6) happened to be written down incorrectly. The accurate trace looks like:

ρ	tr_ρ
$\begin{bmatrix} \mu & \nu & \rho \\ \sigma \end{bmatrix}$	$\frac{1}{8} [(\mu)(\nu)(\rho)(\sigma) + \{(\mu\nu)(\rho)\}(\sigma) - (\mu\sigma)(\nu)(\rho) - (\mu\sigma)(\nu\rho) - (\mu\sigma\{\nu\}(\rho)) + (\mu\{\nu\rho\})(\sigma) - (\mu\sigma\{\nu\rho\})]$

3.2 Lie algebra conventions

The generators of the $u(N)$ Lie algebra are $N \times N$ hermitian matrices $T_\mu = (T_0, T_a)$. In general, we use the Greek letters $\mu, \nu, \rho, \sigma \in \{0, 1, \dots, N^2 - 1\}$ to label the $u(N)$ Lie algebra generators, and we use the Latin letters $a, b, c, d \in \{1, \dots, N^2 - 1\}$ to label the $\mathfrak{su}(N)$ Lie algebra generators, where $\mu \in \{0, a\}$, $\nu \in \{0, b\}$, $\rho \in \{0, c\}$ and $\sigma \in \{0, d\}$.

$\mathfrak{su}(N)$ Lie algebra relations

Here, we present some essential relations between the $\mathfrak{su}(N)$ Lie algebra generators T_a . We would like to refer the interested readers to the original document [3], where the proof of all the listed relations can be found except (36), which seems to be a new relation firstly derived here. T_a is the set of $N \times N$ hermitian traceless matrices, thus

$$\text{tr } T_a = 0. \quad (26)$$

We stick to the following normalization convention in the defining representation of $\mathfrak{su}(N)$:

$$\text{tr}(T_a T_b) = \frac{1}{2} \delta_{ab}. \quad (27)$$

The generators satisfy the commutation relation

$$\begin{aligned} [T_a, T_b] &\equiv T_a T_b - T_b T_a = i f_{abc} T_c, \quad \text{whence} \\ f_{abc} &= 2i \text{tr}([T_a, T_b] T_c), \end{aligned} \quad (28)$$

where f_{abc} is a totally antisymmetric third rank tensor with respect to the interchange of any pair of indices.

The generators satisfy the anticommutation relation

$$\begin{aligned} \{T_a, T_b\} &\equiv T_a T_b + T_b T_a = \frac{1}{N} \delta_{ab} \mathbb{1} + d_{abc} T_c, \quad \text{whence} \\ d_{abc} &= 2 \text{tr}(\{T_a, T_b\} T_c), \end{aligned} \quad (29)$$

where d_{abc} is a totally symmetric third rank tensor with respect to the interchange of any pair of indices.

For the antisymmetric f_{abc} and the symmetric d_{abc} tensors the following identities are valid:

$$d_{aab} = 0, \quad (30)$$

$$f_{abc} d_{abd} = 0, \quad (31)$$

$$f_{abc} f_{abd} = N \delta_{cd}, \quad (32)$$

$$d_{abc} d_{abd} = \frac{N^2 - 4}{N} \delta_{cd}. \quad (33)$$

Starting from the identity

$$f_{abe} f_{cde} = \frac{2}{N} (\delta_{ac} \delta_{bd} - \delta_{ad} \delta_{bc}) + d_{ace} d_{bde} - d_{bce} d_{ade}, \quad (34)$$

we derive the new relation. Multiplying the previous equation with $d_{bfc} d_{fda}$ and in light of the relations (30,33), we get

$$\begin{aligned} f_{eab} f_{ecd} d_{bfc} d_{fda} &= \frac{2}{N} \frac{N^2 - 4}{N} (N^2 - 1) + \\ &+ d_{ace} d_{bde} d_{bfc} d_{fda} - \frac{N^2 - 4}{N} \frac{N^2 - 4}{N} (N^2 - 1). \end{aligned} \quad (35)$$

By the interchange of the indices $b \leftrightarrow c$ on the right side, we obtain the new formula

$$\begin{aligned} (f_{eab} f_{ecd} - d_{eab} d_{ecd}) d_{bfc} d_{fda} &= \\ &= - \frac{N^2 - 4}{N^2} (N^2 - 1) (N^2 - 6), \end{aligned} \quad (36)$$

which will be utilized in subsection 4.2.

The trace of the product of three generators looks like

$$\text{tr}(T_a T_b T_c) = \frac{1}{4} (d_{abc} + i f_{abc}), \quad (37)$$

and the trace of the product of four generators is

$$\begin{aligned} \text{tr}(T_a T_b T_c T_d) &= \frac{1}{4N} \delta_{ab} \delta_{cd} + \frac{1}{8} (d_{abe} d_{cde} - f_{abe} f_{cde} + \\ &+ i f_{abe} d_{cde} + i f_{cde} d_{abe}). \end{aligned} \quad (38)$$

$u(N)$ Lie algebra relations

We would like to complete the relations of the $\mathfrak{su}(N)$ Lie algebra with the case when we consider the 0th generator of the $u(N)$ Lie algebra: the hermitian non-traceless $T_0 \sim \mathbb{1}$. To find the proportionality coefficient we recall the normalization convention (27), which we extend for the $u(N)$ generators:

$$\text{tr}(T_\mu T_\nu) = \frac{1}{2} \delta_{\mu\nu}. \quad (39)$$

Applying the previous equation for $T_0 T_0$, we conclude that $T_0 = \frac{1}{\sqrt{2N}} \mathbb{1}$.

Our aim for the following calculations is to generalize the trace formulae (26,27,37,38) from the previous section to the case of $u(N)$ generators by examining the particular cases, when at least one of the generators happen to be T_0 , separately. These identities will be extremely useful when one endeavours to derive the relations in the next section.

Generalizing (26), we get

$$\text{tr} T_\mu = \delta_{0\mu} \text{tr} T_0 + \text{tr} T_a = \sqrt{\frac{N}{2}} \delta_{0\mu} . \quad (40)$$

Before proceeding further, let us stress a useful remark. If only one generator, denoted as T , is not equal to T_0 , then by the application of (26) the trace $\text{tr}(T_\mu \dots T \dots T_\sigma)$ is zero:

$$\text{tr}(T_\mu \dots T \dots T_\sigma) \sim \text{tr}(\mathbb{1} \dots T \dots \mathbb{1}) = \text{tr}(T) = 0 . \quad (41)$$

Generalizing (27), we get

$$\text{tr}(T_\mu T_\nu) = \delta_{0\mu} \delta_{0\nu} \text{tr} T_0^2 + 0 + \text{tr}(T_a T_b) = \frac{1}{2} \delta_{0\mu} \delta_{0\nu} + \frac{1}{2} \delta_{ab} . \quad (42)$$

Generalizing (37), we get

$$\begin{aligned} \text{tr}(T_\mu T_\nu T_\rho) &= \delta_{0\mu} \delta_{0\nu} \delta_{0\rho} \text{tr} T_0^3 + 0 + [\delta_{0\mu} \text{tr}(T_0 T_b T_c) + \delta_{0\nu} \text{tr}(T_0 T_a T_c) + \delta_{0\rho} \text{tr}(T_0 T_a T_b)] + \text{tr}(T_a T_b T_c) = \\ &= \frac{1}{2\sqrt{2N}} \delta_{0\mu} \delta_{0\nu} \delta_{0\rho} + \frac{1}{2\sqrt{2N}} (\delta_{0\mu} \delta_{bc} + \delta_{0\nu} \delta_{ac} + \delta_{0\rho} \delta_{ab}) + \frac{1}{4} (d_{abc} + i f_{abc}) . \end{aligned} \quad (43)$$

Generalizing (38), we get

$$\begin{aligned} \text{tr}(T_\mu T_\nu T_\rho T_\sigma) &= \delta_{0\mu} \delta_{0\nu} \delta_{0\rho} \delta_{0\sigma} \text{tr} T_0^4 + 0 + [\delta_{0\mu} \delta_{0\nu} \text{tr}(T_0^2 T_c T_d) + \delta_{0\mu} \delta_{0\rho} \text{tr}(T_0^2 T_b T_d) + \delta_{0\mu} \delta_{0\sigma} \text{tr}(T_0^2 T_b T_c) + \\ &\quad + \delta_{0\nu} \delta_{0\sigma} \text{tr}(T_0^2 T_a T_c) + \delta_{0\nu} \delta_{0\rho} \text{tr}(T_0^2 T_a T_d) + \delta_{0\rho} \delta_{0\sigma} \text{tr}(T_0^2 T_a T_b)] + [\delta_{0\mu} \text{tr}(T_0 T_b T_c T_d) + \\ &\quad + \delta_{0\nu} \text{tr}(T_0 T_a T_c T_d) + \delta_{0\rho} \text{tr}(T_0 T_a T_b T_d) + \delta_{0\sigma} \text{tr}(T_0 T_a T_b T_c)] + \text{tr}(T_a T_b T_c T_d) = \\ &= \frac{1}{4N} \delta_{0\mu} \delta_{0\nu} \delta_{0\rho} \delta_{0\sigma} + \frac{1}{4N} [\delta_{0\mu} \delta_{0\nu} \delta_{cd} + \delta_{0\mu} \delta_{0\rho} \delta_{bd} + \delta_{0\mu} \delta_{0\sigma} \delta_{bc} + \delta_{0\nu} \delta_{0\sigma} \delta_{ac} + \delta_{0\nu} \delta_{0\rho} \delta_{ad} + \\ &\quad + \delta_{0\rho} \delta_{0\sigma} \delta_{ab}] + \frac{1}{4\sqrt{2N}} [\delta_{0\mu} (d_{bcd} + i f_{bcd}) + \delta_{0\nu} (d_{acd} + i f_{acd}) + \delta_{0\rho} (d_{abd} + i f_{abd}) + \\ &\quad + \delta_{0\sigma} (d_{abc} + i f_{abc})] + \left[\frac{1}{4N} \delta_{ab} \delta_{cd} + \frac{1}{8} (d_{abe} d_{cde} - f_{abe} f_{cde} + i f_{abe} d_{cde} + i f_{cde} d_{abe}) \right] . \end{aligned} \quad (44)$$

4 Perturbative expansion of the model

In this section, we present the perturbative expansion of the kinetic term firstly in the order of $\mathcal{O}(\beta)$ and secondly in the order of $\mathcal{O}(\beta^2)$. This means we intend to find the sums in relations (24) and (25) profiting from the tables (B.4), (B.5), (B.6) and (B.8) in [1].

4.1 Perturbative expansion in the order of $\mathcal{O}(\beta)$

Benefiting from the above mentioned references, we calculate $I_{\mu\nu}$ from (24) applying the strategy mentioned in section 3.1

$$\begin{aligned} I_{\mu\nu} &= \sum_\rho \frac{1}{\dim \rho} \text{tr}_\rho (\Lambda \otimes \Lambda) \text{tr}_\rho (T_\mu \otimes T_\nu) = \\ &= \frac{1}{\dim \boxed{\mu} \boxed{\nu}} \text{tr}_{\boxed{\mu} \boxed{\nu}} (\Lambda \otimes \Lambda) \text{tr}_{\boxed{\mu} \boxed{\nu}} (T_\mu \otimes T_\nu) + \frac{1}{\dim \boxed{\mu} \boxed{\nu}} \text{tr}_{\boxed{\mu} \boxed{\nu}} (\Lambda \otimes \Lambda) \text{tr}_{\boxed{\mu} \boxed{\nu}} (T_\mu \otimes T_\nu) = \\ &= \frac{1}{\frac{N^2+N}{2}} \frac{1}{2} (\text{tr}^2 \Lambda + \text{tr} \Lambda^2) \frac{1}{2} [\text{tr}(T_\mu) \text{tr}(T_\nu) + \text{tr}(T_\mu T_\nu)] + \frac{1}{\frac{N^2-N}{2}} \frac{1}{2} (\text{tr}^2 \Lambda - \text{tr} \Lambda^2) \frac{1}{2} [\text{tr}(T_\mu) \text{tr}(T_\nu) - \text{tr}(T_\mu T_\nu)] . \end{aligned} \quad (45)$$

Utilizing the trace formulae (40,42) yields

$$I_{\mu\nu} = \left[\frac{\delta_{0\mu} \delta_{0\nu}}{2N} - \frac{\delta_{ab}}{2N(N^2-1)} \right] \text{tr}^2 \Lambda + \frac{\delta_{ab}}{2(N^2-1)} \text{tr} \Lambda^2 = \frac{\delta_{0\mu} \delta_{0\nu}}{2N} \text{tr}^2 \Lambda + \frac{\delta_{ab}}{2(N^2-1)} \left(\text{tr} \Lambda^2 - \frac{1}{N} \text{tr}^2 \Lambda \right) . \quad (46)$$

4.2 Perturbative expansion in the order of $\mathcal{O}(\beta^2)$

By a similar analysis to (45), we calculate $I_{\mu\nu\rho\sigma}$ from (25) utilizing the trace formulae (40,42,43,44). After all the dust settles, one gets

$$\begin{aligned}
I_{\mu\nu\rho\sigma} &= \sum_{\rho} \frac{1}{\dim \rho} \text{tr}_{\rho} (\Lambda \otimes \Lambda \otimes \Lambda \otimes \Lambda) \text{tr}_{\rho} (T_{\mu} \otimes T_{\nu} \otimes T_{\rho} \otimes T_{\sigma}) = \\
&= \frac{1}{4N(N^2(N^2-7)^2-36)} [2(4-N^2)(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}) + \\
&\quad + (N^3+N)(d_{cde}(d_{abe} + if_{abe}) + d_{bde}(d_{ace} + if_{ace}) + d_{bce}(d_{ade} + if_{ade}))] \text{tr} \Lambda^4 + \\
&+ \frac{1}{4N^2(N^2(N^2-7)^2-36)} [(N^2-6)(N^2-4)(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}) - \\
&\quad - N(2N^2-3)(d_{cde}(d_{abe} + if_{abe}) + d_{bde}(d_{ace} + if_{ace}) + d_{bce}(d_{ade} + if_{ade}))] \text{tr}^2 \Lambda^2 + \\
&+ \frac{1}{4N^2(N^2(N^2-7)^2-36)} [8(N^2-4)(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}) + \\
&\quad + \sqrt{2}N^{5/2}(N^2-9)(\delta_{0\mu}d_{bcd} + \delta_{0\nu}d_{acd} + \delta_{0\rho}d_{abd} + \delta_{0\sigma}d_{abc}) - \\
&\quad - 4N(N^2+1)(d_{cde}(d_{abe} + if_{abe}) + d_{bde}(d_{ace} + if_{ace}) + d_{bce}(d_{ade} + if_{ade}))] \text{tr} \Lambda \text{tr} \Lambda^3 + \\
&+ \frac{1}{4N(N^2(N^2-7)^2-36)} [2(4-N^2)(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}) + \\
&\quad + (N^2-9)(N^2-4)(\delta_{0\mu}\delta_{0\nu}\delta_{cd} + \delta_{0\mu}\delta_{0\rho}\delta_{bd} + \delta_{0\mu}\delta_{0\sigma}\delta_{bc} + \delta_{0\nu}\delta_{0\rho}\delta_{ad} + \delta_{0\nu}\delta_{0\sigma}\delta_{ac} + \delta_{0\rho}\delta_{0\sigma}\delta_{ab}) + \\
&\quad + 3\sqrt{2}N^{1/2}(9-N^2)(\delta_{0\mu}d_{bcd} + \delta_{0\nu}d_{acd} + \delta_{0\rho}d_{abd} + \delta_{0\sigma}d_{abc}) + \\
&\quad + 10N(d_{cde}(d_{abe} + if_{abe}) + d_{bde}(d_{ace} + if_{ace}) + d_{bce}(d_{ade} + if_{ade}))] \text{tr}^2 \Lambda \text{tr} \Lambda^2 + \\
&+ \frac{1}{4N^2(N^2(N^2-7)^2-36)} [(N^2-4)(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}) + \\
&\quad + (N^2-9)(-N^2+4)(\delta_{0\mu}\delta_{0\nu}\delta_{cd} + \delta_{0\mu}\delta_{0\rho}\delta_{bd} + \delta_{0\mu}\delta_{0\sigma}\delta_{bc} + \delta_{0\nu}\delta_{0\rho}\delta_{ad} + \delta_{0\nu}\delta_{0\sigma}\delta_{ac} + \delta_{0\rho}\delta_{0\sigma}\delta_{ab}) + \\
&\quad + 2\sqrt{2}N^{1/2}(N^2-9)(\delta_{0\mu}d_{bcd} + \delta_{0\nu}d_{acd} + \delta_{0\rho}d_{abd} + \delta_{0\sigma}d_{abc}) + \\
&\quad + \delta_{0\mu}\delta_{0\nu}\delta_{0\rho}\delta_{0\sigma}(N^2-9)(N^2-4)(N^2-1) - \\
&\quad - 5N(d_{cde}(d_{abe} + if_{abe}) + d_{bde}(d_{ace} + if_{ace}) + d_{bce}(d_{ade} + if_{ade}))] \text{tr}^4 \Lambda. \tag{47}
\end{aligned}$$

We put the result $I_{\mu\nu\rho\sigma}$ under five different tests to prove its validity in the cases when we know the exact result. In section 3.1, we have already made an acquaintance with the relations

$$I_{\mu\nu\rho\sigma} = \frac{1}{16} \int [dU] \Phi_{\mu} \Phi_{\nu} \Phi_{\rho} \Phi_{\sigma} \quad \text{and} \quad \Phi_{\mu} T_{\mu} = \Phi = U \Lambda U^{\dagger}.$$

Using them, utilizing the cyclic property of trace and realizing that the integral over the Dyson measure $[dU]$ is normalized, we get:

$$\begin{aligned}
\text{tr}(T_{\mu} T_{\nu} T_{\rho} T_{\sigma}) I_{\mu\nu\rho\sigma} &= \frac{1}{16} \int [dU] \text{tr}(T_{\mu} T_{\nu} T_{\rho} T_{\sigma}) \Phi_{\mu} \Phi_{\nu} \Phi_{\rho} \Phi_{\sigma} = \frac{1}{16} \int [dU] \text{tr}(\Phi \Phi \Phi \Phi) = \\
&= \frac{1}{16} \int [dU] \text{tr}(U \Lambda U^{\dagger} U \Lambda U^{\dagger} U \Lambda U^{\dagger} U \Lambda U^{\dagger}) = \frac{1}{16} \text{tr} \Lambda^4 \int [dU] = \frac{1}{16} \text{tr} \Lambda^4.
\end{aligned}$$

One can prove the following identities in a similar way as the first one was done:

$$\begin{aligned} \text{tr}(T_\mu T_\nu T_\rho) \text{tr}(T_\sigma) I_{\mu\nu\rho\sigma} &= \frac{1}{16} \text{tr}\Lambda^3 \text{tr}\Lambda & \text{tr}(T_\mu T_\nu) \text{tr}(T_\rho T_\sigma) I_{\mu\nu\rho\sigma} &= \frac{1}{16} \text{tr}^2\Lambda^2 \\ \text{tr}(T_\mu T_\nu) \text{tr}(T_\rho) \text{tr}(T_\sigma) I_{\mu\nu\rho\sigma} &= \frac{1}{16} \text{tr}\Lambda^2 \text{tr}^2\Lambda & \text{tr}(T_\mu) \text{tr}(T_\nu) \text{tr}(T_\rho) \text{tr}(T_\sigma) I_{\mu\nu\rho\sigma} &= \frac{1}{16} \text{tr}^4\Lambda. \end{aligned}$$

With the help of the earned result (47) we evaluated all the left sides of the previous equations. Using the trace formulae (40,42,43,44), we contracted them with $I_{\mu\nu\rho\sigma}$, and with the help of the listed $\mathfrak{su}(N)$ identities from 3.2, we obtained the corresponding right side to the examined left side. These necessary, lengthy and sweaty checks also verified the correctness of (47).

5 Kinetic term effective action of the models

In the present work, we will deal with the fuzzy sphere S_F^2 and the Grosse-Wulkenhaar matrix model. In both of the cases, we will determine the effective action of the kinetic term $S_{\text{kin}}^{\text{eff}}$ from equation (19) and examine their large N limit. The models are defined through their kinetic term, which means they differ in the eigenvalues k_μ coming from the equation $\mathcal{K}T_\mu = k_\mu T_\mu$, where the eigenvalues k_μ are in both cases, in principle, known. In the beginning, we can do our calculations generally, without further specializing the values of k_μ .

Combining equations (19), (21) and (46), we get

$$S_1 = I_1 = 2k_\mu \delta_{\mu\nu} \left[\frac{\delta_{0\mu} \delta_{0\nu}}{2N} \text{tr}^2\Lambda + \frac{\delta_{ab}}{2(N^2-1)} \left(\text{tr}\Lambda^2 - \frac{1}{N} \text{tr}^2\Lambda \right) \right] = \frac{k_0}{N} \text{tr}^2\Lambda + \frac{\Sigma_1}{N^2-1} \left(\text{tr}\Lambda^2 - \frac{1}{N} \text{tr}^2\Lambda \right), \quad (48)$$

where we have defined

$$\Sigma_1 = \sum_{a=1}^{N^2-1} k_a \quad \text{and} \quad \Sigma_2 = \sum_{a=1}^{N^2-1} k_a^2. \quad (49)$$

Equation (48) is the general form of the effective action of arbitrary kinetic term up to the order $\mathcal{O}(\beta)$.

Combining equations (25) and (47), we get

$$\begin{aligned} I_2 &= 4k_\mu \delta_{\mu\nu} k_\rho \delta_{\rho\sigma} I_{\mu\nu\rho\sigma} = \\ &= \frac{1}{N(N^2(N^2-7)^2-36)} \left[-2(N^2-4)(\Sigma_1^2+2\Sigma_2) + N(N^2+1)(k_a d_{aae} k_b d_{bbe} + 2k_a k_b d_{abe} d_{abe}) \right] \text{tr}\Lambda^4 + \\ &+ \frac{1}{N^2(N^2(N^2-7)^2-36)} \left[(N^2-6)(N^2-4)(\Sigma_1^2+2\Sigma_2) - \right. \\ &\quad \left. - N(2N^2-3)(k_a d_{aae} k_b d_{bbe} + 2k_a k_b d_{abe} d_{abe}) \right] \text{tr}^2\Lambda^2 + \\ &+ \frac{4}{N^2(N^2(N^2-7)^2-36)} \left[2(N^2-4)(\Sigma_1^2+2\Sigma_2) - \right. \\ &\quad \left. - N(N^2+1)(k_a d_{aae} k_b d_{bbe} + 2k_a k_b d_{abe} d_{abe}) \right] \text{tr}\Lambda \text{tr}\Lambda^3 + \\ &+ \frac{2}{N(N^2(N^2-7)^2-36)} \left[(N^2-9)(N^2-4)k_0 \Sigma_1 - (N^2-4)(\Sigma_1^2+2\Sigma_2) + \right. \\ &\quad \left. + 5N(k_a d_{aae} k_b d_{bbe} + 2k_a k_b d_{abe} d_{abe}) \right] \text{tr}^2\Lambda \text{tr}\Lambda^2 + \\ &+ \frac{1}{N^2(N^2(N^2-7)^2-36)} \left[k_0^2(N^2-9)(N^2-4)(N^2-1) - 2(N^2-9)(N^2-4)k_0 \Sigma_1 + \right. \\ &\quad \left. + (N^2-4)(\Sigma_1^2+2\Sigma_2) - 5N(k_a d_{aae} k_b d_{bbe} + 2k_a k_b d_{abe} d_{abe}) \right] \text{tr}^4\Lambda, \end{aligned} \quad (50)$$

where the terms $k_a d_{aae} k_b d_{bbe}$ and $k_a k_b d_{abe} d_{abe}$ are understood in the way that the summations go over the triplet of the indices a and b and the pair of the indices e , thus the $\mathfrak{su}(N)$ identities like (33) cannot be utilized. With the help of the equations (48) and (50) we can write down the general form of the effective action $S_2 = \frac{1}{2}(I_1^2 - I_2)$ of arbitrary kinetic term in the order of $\mathcal{O}(\beta^2)$.

In the present cases, the kinetic terms are defined through commutation relations (13,16), and thus they depend only on T_a , since for $T_0 \sim \mathbb{1}$ the commutators vanish and therefore $k_0 = 0$.

The main goal of this paper is to provide the effective action of our models, which are solved in large N limit to reach our final destination – the phase diagrams. Examining this asymptotic behaviour, the discrete set of eigenvalues λ_i of matrix Λ transforms into continuous function in the way

$$\lambda_i \xrightarrow{N \rightarrow \infty} \lambda(i/N) \equiv \lambda(x) \quad \text{where} \quad 0 < x \leq 1. \quad (51)$$

This modification transforms the traces of different powers of the Λ matrices to integrals in the way

$$\text{tr} \Lambda^j = \sum_{i=1}^N \lambda_i^j \xrightarrow{N \rightarrow \infty} N \int_0^1 dx \lambda(x)^j \equiv N c_j. \quad (52)$$

For further convenience, we introduce the symmetrized moments t_n as

$$t_n = \frac{1}{N} \text{tr} \left(\Phi - \frac{\mathbb{1}}{N} \text{tr}(\Phi) \right)^n. \quad (53)$$

Recalling $\Phi = U \Lambda U^\dagger$ and the previously described rescaling in the limit $N \rightarrow \infty$, we get the first t_n 's explicitly

$$\begin{aligned} t_2 &= c_2 - c_1^2, \\ t_3 &= c_3 - 3c_1 c_2 + 2c_1^3, \\ t_4 &= c_4 - 4c_3 c_1 + 6c_2 c_1^2 - 3c_1^4. \end{aligned} \quad (54)$$

5.1 Effective action of the fuzzy sphere

The eigenvalues for the S_F^2 from equation (13) give the sums (49) considering the correspondent degeneracy of eigenspaces:

eracy of eigenspaces:

$$\begin{aligned} \Sigma_1 &= \sum_{l=1}^{N-1} (2l+1) \left[\frac{1}{2} l(l+1) \right] = \frac{1}{4} (N-1) N^2 (N+1) \\ \Sigma_2 &= \sum_{l=1}^{N-1} (2l+1) \left[\frac{1}{2} l(l+1) \right]^2 = \\ &= \frac{1}{12} (N-1)^2 N^2 (N+1)^2. \end{aligned} \quad (55)$$

The terms in (50) containing the symmetric tensor d_{abe} were found utilizing numerical calculations:

$$\begin{aligned} k_a d_{aae} k_b d_{bbe} &= 0 \\ k_a k_b d_{abe} d_{abe} &= \frac{1}{48} N (3N^6 - 14N^4 + 7N^2 + 4). \end{aligned} \quad (56)$$

This was done as follows. We have generated matrices T_μ and coefficients d_{abc} explicitly for a fixed value of N and calculated the above sums. We have repeated this process for increasing N from 2 to 10 and then fitted the data with a polynomial, obtaining the given result. As a check of this method, we have calculated the sums in (55) in the same way, obtaining the expected results. Sums (56) could be in principle calculated analytically using advantage of the specific form of the fuzzy sphere kinetic term (13). However, since this would no longer be the case in the section 5.2, we did not proceed this way.

Substituting the sums (55,56) to equations (48) and (50) and taking into consideration the above mentioned rescaling, we thus find S_1 and S_2 from (19) for finite N (although the traces have already gone through the rescaling) as

$$\begin{aligned} S_1 &= \frac{1}{4} N^3 (c_2 - c_1^2) \\ S_2 &= \frac{1}{48} N^2 \left(-N^2 c_1^4 + 2N^2 c_1^2 c_2 - 4c_1 c_3 - \right. \\ &\quad \left. - (N^2 - 3) c_2^2 + c_4 \right). \end{aligned} \quad (57)$$

The effective action of the kinetic term for the S_F^2 has already been calculated in [2] for finite N up to the order of $\mathcal{O}(\beta^3)$. The comparison of our result with their known result and the subsequent recalculation of their result exposed a typo in this paper [2]: on page 7 the factors corresponding to the irreducible representations of four fundamental representations should contain 8 instead of 20 in the numerators. This error carries over the formula for $\mathcal{S}_2$ on the page 8,

too. However, for the large N limit, the following relations are again correct.

Considering the limit $N \rightarrow \infty$, we get the kinetic term effective action from (19)

$$\begin{aligned} S_{\text{kin}}^{\text{eff}} &= \frac{1}{4}N^2(c_2 - c_1^2) - \frac{1}{48}N^2(c_2 - c_1^2)^2 = \\ &= \frac{1}{4}N^2t_2 - \frac{1}{48}N^2t_2^2, \end{aligned} \quad (58)$$

where we have put $\beta = 1$, and the continuous function of eigenvalues $\lambda(x)$ was rescaled in the way $\lambda(x) \rightarrow \lambda(x)/\sqrt{N}$. Note that this rescaling in the kinetic term alters the potential terms in the exponent in (12). The obtained relation (58) is in agreement with [2]. This also confirms the validity of our results (56).

5.2 Effective action of the Grosse-Wulkenhaar model

Using the same numerical process as we did to obtain (56), we get the necessary expressions for the GW model kinetic term (16)

$$\begin{aligned} \Sigma_1 &= N^2(N-1) \\ \Sigma_2 &= \frac{1}{6}N^2(10N^2 - 21N + 11), \end{aligned} \quad (59)$$

additionally,

$$\begin{aligned} k_a d_{aac} k_b d_{bbc} &= \frac{1}{6}N^3(N-1)(N-2) \\ k_a k_b d_{abc} d_{abc} &= \frac{1}{120}N(13N^4 - 270N^3 - 260N^2 + \\ &\quad + 720N - 320). \end{aligned} \quad (60)$$

In an analogous way to the fuzzy sphere, we thus find $S_{\text{kin}}^{\text{eff}}$ from (19) for $N \rightarrow \infty$ as

$$\begin{aligned} S_{\text{kin}}^{\text{eff}} &= N^2(c_2 - c_1^2) - \frac{1}{120}N^2(565c_1^4 - 1130c_1^2c_2 + \\ &\quad + 388c_1c_3 + 274c_2^2 - 97c_4) = \\ &= N^2t_2 - \frac{1}{120}N^2(274t_2^2 - 97t_4), \end{aligned} \quad (61)$$

where we have put $\beta = 1$, and the rescaling of the continuous function of eigenvalues $\lambda(x)$ was not required.

In the previous numerical simulations [8] for the GW model (15) appeared a stable asymmetric solution,

which could not be explained analytically considering only the harmonic oscillator potential and the ordinary potential terms [6]. This anticipates that the asymmetric solution is present only due to the kinetic term. Since the $S_{\text{kin}}^{\text{eff}}$ includes the trace of the odd powers of matrices Λ , we are optimistic that the kinetic term is indeed responsible for the asymmetric behaviour.

6 Conclusion

To conclude, let us summarize our results. In this paper, we were dealing with scalar field theory defined on fuzzy spaces, and as the first notable result, we derived the second order term in the expansion of an arbitrary kinetic term. This was achieved by using numerous relations and identities within $\mathfrak{su}(N)$ Lie algebra, and we derived a presumably new one (36).

We tested the obtained formula from the expansion for that cases where we explicitly know the results, and in all of the cases we got total agreement. During the calculations we revealed some typos from previously published papers [1, 2]. As the second notable result, we utilized our formula to derive the kinetic term effective action up to the second order for the Grosse-Wulkenhaar matrix model. This model was previously examined through numerical simulations, and we hope that now we dispose enough knowledge about the kinetic term to explain analytically the emerging asymmetric solution.

Looking into the future, we intend to apply the saddle-point approximation for the effective action of the GW model in order to extract from it the related phase diagram. The brunt of the work has been already done in previous documents, but there are terms in (61) which have not probed by saddle-point approximation hitherto – most importantly the c_1c_3 multitrace term. An other future project is to tackle the two-point correlation functions $\langle \Phi(x)\Phi(y) \rangle$ and obtain the effective matrix model for arbitrary kinetic term, which enables to check the numerical results analytically and similar $\mathfrak{su}(N)$ techniques will be used as this work.

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