

Non-diagonal critical central sections of the cube

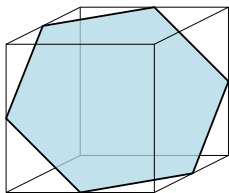
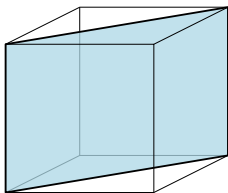
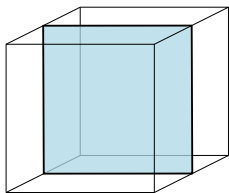
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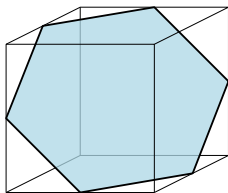
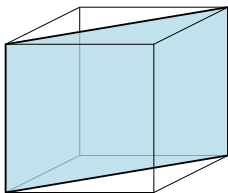
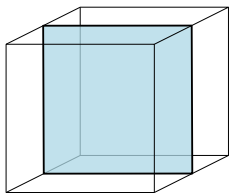
Joint work with Gergely Ambrus

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Szeged, 5 July 2024

Introduction



Introduction



Volume of central hyperplane sections of $Q_n = \left[-\frac{1}{2}, \frac{1}{2}\right]^n$ as a function of its normal vector:

$$\sigma(v) = \text{Vol}_{n-1}(Q_n \cap v^\perp)$$

$\sigma(v)$ is invariant under scalings of v by a non-zero factor, and by embeddings in higher dimensions.

Previous results

- $\sigma(v)$ is calculable with [Pólya's \(1913\)](#) integral formula.
- Minimal sections are parallel to a facet of Q_n , their volume is 1 ([Hadwiger \(1972\)](#)).
- Maximal sections are orthogonal to the diagonal of a 2-dimensional face of Q_n , their volume is $\sqrt{2}$ ([Ball \(1986\)](#)).
- Noncentral and lower dimensional sections. ([Ball](#), [Ivanov](#), [König](#), [Moody](#), [Stone](#), [Vaaler](#), [Zach](#), [Zvavitch](#))
- Sections of the regular simplex and ℓ_p unit balls. ([Chasapis](#), [Dirksen](#), [Meyer](#), [Nayar](#), [Pajor](#), [Tkocz](#), [Webb](#))

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Definition

$v \in S^{n-1}$ is a **critical direction** if it is a critical point of the function $\sigma(v)$ on S^{n-1} . Then $Q_n \cap v^\perp$ is a **critical section**.

Locally extremal sections are defined similarly.

Definition

Unit vectors parallel to the diagonal of a k -dimensional face of Q_n are called *k -diagonal directions*. Corresponding sections are *k -diagonal sections*.

Up to permutation of coordinates and change of signs, they have the form of

$$d_{n,k} := \frac{1}{\sqrt{k}} (\underbrace{1, \dots, 1}_k, \underbrace{0, \dots, 0}_{n-k}).$$

- Based on [Hensley's \(1979\)](#) asymptotic formula, if $k \approx n$ then

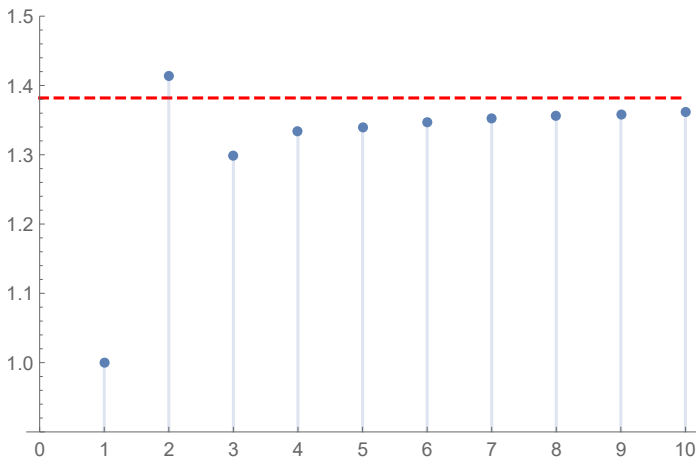
$$\sigma(d_{n,k}) \approx \sqrt{\frac{6}{\pi}}.$$

- [Bartha, Fodor and González Merino \(2020\)](#) showed that for fixed n the sequence of k -diagonal sections is strictly monotone increasing for $k \geq 3$.

Diagonal directions

Definition

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Maximality of diagonal sections

Theorem (Pournin (2023))

For all $k \geq 3$ k -diagonal sections are strictly locally maximal among central sections of Q_n for each $n \geq 4$.

We provided an alternative proof in the special case $k = n$.

Maximality of diagonal sections

Theorem (Pournin (2023))

For all $k \geq 3$ k -diagonal sections are strictly locally maximal among central sections of Q_n for each $n \geq 4$.

Theorem

The main diagonal section $Q_n \cap 1_n^\perp$ has strictly locally maximal volume among central sections of Q_n for each $n \geq 4$.

Our proof uses Lagrange multiplier methods. This requires first to show that diagonal directions are critical.

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The main diagonal section $Q_n \cap 1_n^\perp$ has strictly locally maximal volume among central sections of Q_n for each $n \geq 4$.

Statement (Ambrus (2022))

$v \in S^{n-1}$ is a critical direction if and only if up to permuting coordinates and changing signs $v = e_1$, or

$$\sigma(v) = \frac{1}{\pi(1 - v_j^2)} \int_{-\infty}^{\infty} \prod_{i \neq j} \operatorname{sinc}(v_i t) \cdot \cos(v_j t) dt$$

holds for each $j = 1, \dots, n$.

Main idea of the proof

Suppose that $v \in S^{n-1}$ is a critical direction. Then v is a stationary point of the Lagrange function

$$\Lambda(v) = \sigma(v) + \frac{\sigma(v)}{2} \cdot (|v|^2 - 1)$$

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Its bordered Hessian matrix is

$$H(\Lambda(v)) = \begin{bmatrix} 0 & 2v_1 & 2v_2 & \dots & 2v_n \\ 2v_1 & & & & \\ 2v_2 & \frac{\partial^2 \sigma}{\partial v_j \partial v_k}(v) + \sigma(v) \cdot \begin{cases} 0, & \text{if } j \neq k \\ 1, & \text{if } j = k \end{cases} & & & \\ \vdots & & & & \\ 2v_n & & & & \end{bmatrix}$$

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The main diagonal direction $d_n = \frac{1}{\sqrt{n}} 1_n$ is critical. Denote with H_m the m th principal minors of $H(\Lambda(d_n))$ ($m = 3, \dots, n$).

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The main diagonal direction $d_n = \frac{1}{\sqrt{n}} 1_n$ is critical. Denote with H_m the m th principal minors of $H(\Lambda(d_n))$ ($m = 3, \dots, n$). If

$$\forall m \quad (-1)^{m-1} H_m > 0$$

then $\sigma(v)$ is strictly locally maximal at d_n respect to $v \in S^{n-1}$.

Further characterization of critical directions

Theorem (Ambrus (2022))

$v = (v_1, \dots, v_n) \in S^{n-1}$ is critical direction if and only if up to permuting coordinates and change of signs $v = d_{n,2}$, or there exists some $\mu > 0$ for which

$$\text{Vol}_{n-1}(\text{conv}(0 \cup (R_k \cap v^\perp))) = \mu(1 - v_k^2)$$

holds true for each $k = 1, \dots, n$, where
 $R_k = \{(q_1, \dots, q_n) \in Q_n : q_k = 1\}$.

Corollary (Ambrus (2022))

For $n = 2, 3$, all critical directions are diagonal. If $n = 4$, then the critical directions are either diagonal or parallel to the vector $(1, 1, 2, 2)$ up to permuting coordinates and changing signs.

Non-diagonal critical sections

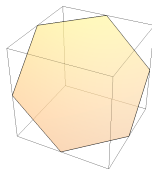
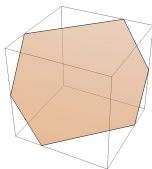
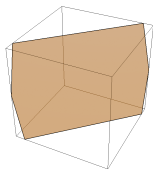
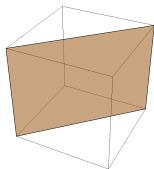
Theorem

For all $n \geq 4$ there exist non-diagonal critical central sections of Q_n whose normal vector is not parallel to any of the coordinate axes.

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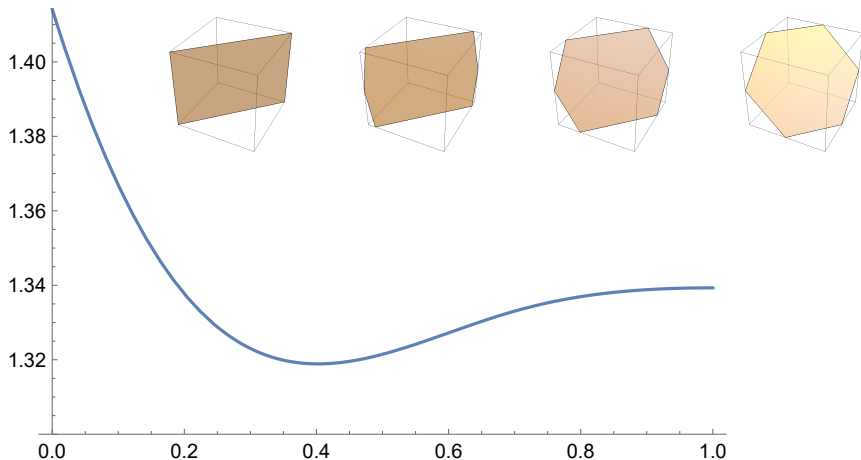
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Non-diagonal critical directions

Consider the following class of vectors:

$$v_{n,k}(a) := (\underbrace{a, \dots, a}_k, \underbrace{b, \dots, b}_{n-k}) \in S^{n-1},$$

where $a \in I_k := \left[0, \frac{1}{\sqrt{k}}\right]$, and $b := b_{n,k}(a) = \sqrt{\frac{1 - ka^2}{n-k}}$.

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Main idea of the proof: there is some $a \in I_2$, for which the vector $v_{n,2}(a)$ is non-diagonal, consists non-zero coordinates and is a critical direction.

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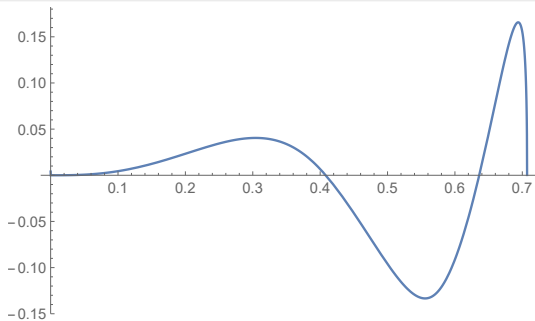
Main idea of the proof: there is some $a \in I_2$, for which the vector $v_{n,2}(a)$ is non-diagonal, consists non-zero coordinates and is a critical direction.

Based on the *Characterization Theorem*, $v_{n,k}(a)$ is critical direction for exactly those a 's, which are zeros of the function

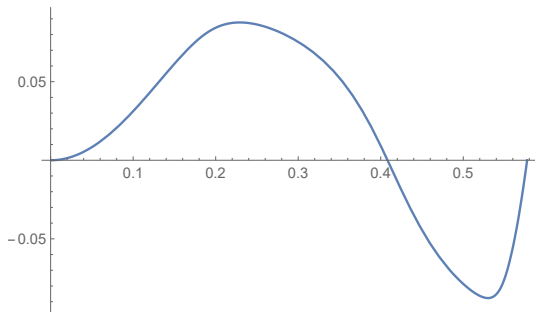
$$\begin{aligned} F_{n,k}(a) := & \frac{1}{1 - a^2} \int_{-\infty}^{\infty} \operatorname{sinc}^{n-k} bt \cdot \operatorname{sinc}^{k-1} at \cdot \cos at \, dt - \\ & - \frac{1}{1 - b^2} \int_{-\infty}^{\infty} \operatorname{sinc}^{n-k-1} bt \cdot \operatorname{sinc}^k at \cdot \cos bt \, dt. \end{aligned}$$

Building up the proof

graph of $F_{6,2}$



graph of $F_{6,3}$



Lemma 1

For each $2 \leq k \leq n-2$, $\frac{1}{\sqrt{k}}$ and $\frac{1}{\sqrt{n}}$ are both zeros of $F_{n,k}$.

Lemma 2

For each $4 \leq k \leq n-2$, $F_{n,k}$ is differentiable I_k . In the case of $k=2,3$, differentiability is true for every compact subinterval of I_k which does not contain the right end point. Moreover, in both cases we have

$$F'_{n,k}\left(\frac{1}{\sqrt{n}}\right) < 0.$$

Lemma 3

For each $n \geq 4$, $F_{n,2}(a) \geq 0$ for every $a \in \left[\gamma_n, \frac{1}{\sqrt{2}}\right]$ where

$$\gamma_n = \sqrt{\frac{n-2}{2n-3}}.$$

Non-diagonal directions are not extremal

- ξ_n is the (first) zero guaranteed by the proof and $w = v_{n,2}(\xi_n)$.

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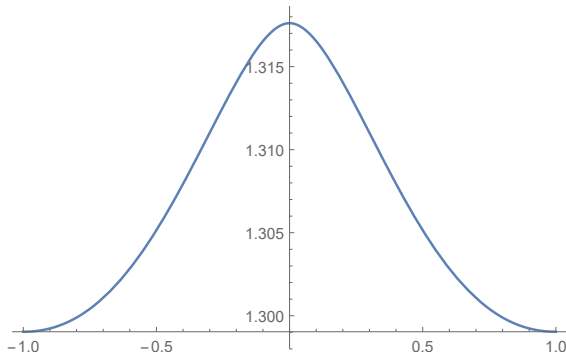
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- Show that the Hessian matrix of the Lagrange function $\tilde{H}$ is not positive definite at w .
- Find a vector q such that $q\tilde{H}q^T < 0$.
- Suitable choice: $q = (1, -1, 0, \dots, 0)$.



Laplace-Pólya integral

Statement

For each $n \geq 2$

$$(n+3)J_{n+2}(0) < (n+2)J_n(0),$$

where

$$J_n(r) = \frac{1}{\pi} \int_{-\infty}^{\infty} \text{sinc}^n t \cdot \cos(rt) \, dt$$

Connection of this integral with other fields:

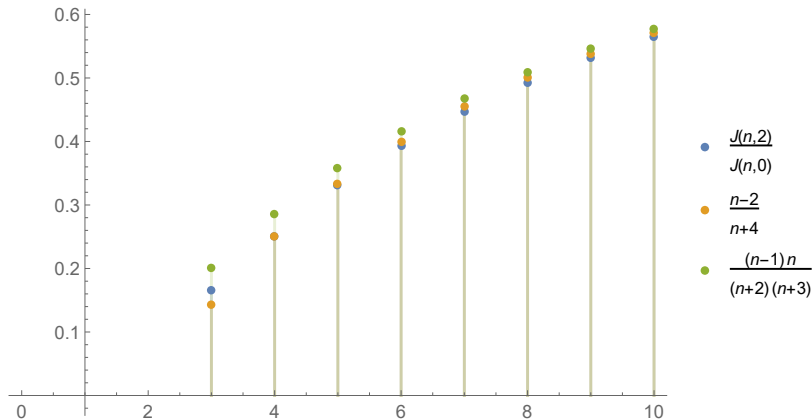
- geometry: volume of hyperplane sections of Q_n
- probability theory: probability density function of $\sum_{i=1}^n X_i$
- combinatorics: recursive formula by [Thompson \(1966\)](#)

$$J_n(r) = \frac{n+r}{2(n-1)} J_{n-1}(r+1) + \frac{n-r}{2(n-1)} J_{n-1}(r-1)$$

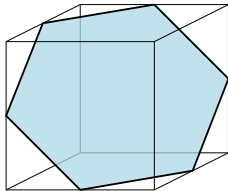
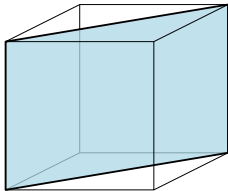
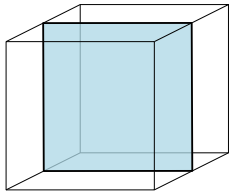
Theorem

Let $n \geq 4$ and r be integers satisfying $-1 \leq r \leq n-2$. Then

$$\frac{J_n(r+2)}{J_n(r)} \leq \frac{(n-r-2)(n-r)(n-r+2)}{(n+r)(n+r+2)(n+r+4)}.$$



Thank you for your attention!



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- Number of zeros of $F_{n,2}$ and so the number of non-diagonal critical directions?
- Asymptotic behaviour of the zeros of $F_{n,2}$?
- Are there other type of critical directions?
- In the case of $k \geq 3$, are unit vectors $v_{n,k}$ not critical directions?

The coordinates of critical directions are determined by the zeros of the function

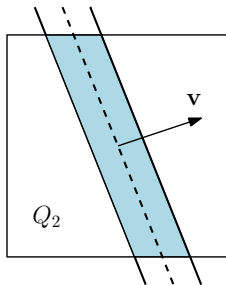
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In small dimensions these are:

4	0,632455	13	0,638774	22	0,639416
5	0,634265	14	0,638893	23	0,639453
6	0,636071	15	0,638998	24	0,639486
7	0,636935	16	0,639081	25	0,639517
8	0,637520	17	0,639156	26	0,639545
9	0,637921	18	0,639222	27	0,639570
10	0,638219	19	0,639278	28	0,639594
11	0,638445	20	0,639329	29	0,639616
12	0,638625	21	0,639375	30	0,639636

Tools from probability theory

Let $X_1, \dots, X_n$ be independent random variables distributed uniformly on $[-1, 1]$. The joint distribution $(X_1, \dots, X_n)$ induces the normalized Lebesgue measure on $2Q_n$. Let $v \in S^{n-1}$.



$$\mathbb{P}\left(\left|\sum_{i=1}^n v_i X_i - r\right| \leq \varepsilon\right) = \frac{1}{2^n} \text{Vol}_n(\mathbf{q} \in 2Q_n : |\langle \mathbf{q}, \mathbf{v} \rangle - r| \leq \varepsilon)$$

$$2f_{\sum_{i=1}^n v_i X_i}(r) = \frac{1}{|v|} s\left(v, \frac{r}{2}\right).$$

The characteristic function of the random variable $\sum_{i=1}^n v_i X_i$ is

$$\varphi_{\sum_{i=1}^n v_i X_i}(t) = \prod_{i=1}^n \text{sinc}(v_i t),$$

where

$$\text{sinc } x = \begin{cases} \frac{\sin x}{x}, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0. \end{cases}$$

Hence $s(v, r)$ is derived by taking the inverse Fourier transform:

$$s(v, r) = \frac{|v|}{\pi} \int_{-\infty}^{\infty} \prod_{i=1}^n \text{sinc}(v_i t) \cdot \cos(2rt) dt.$$

Then the normalized central section $s(v, 0)$ is

$$\sigma(v) = \frac{|v|}{\pi} \int_{-\infty}^{\infty} \prod_{i=1}^n \text{sinc}(v_i t) dt = s(v, 0).$$

Lagrange multiplier method

If $v \in S^{n-1}$ is a critical direction, then

$$\frac{\partial}{\partial v_i} \sigma(v) = -\sigma(v) \cdot v_i.$$

Accordingly, v is a stationary point of the Lagrange function

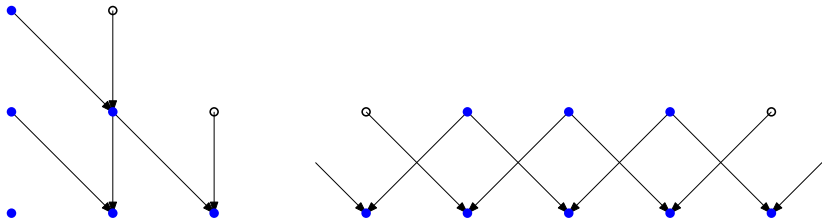
$$\Lambda(v) = \sigma(v) + \tilde{\lambda}(|v|^2 - 1)$$

where

$$\tilde{\lambda} = \frac{\sigma(v)}{2}$$

is the Lagrange multiplier.

Connection with Eulerian numbers



Recursive formula of Eulerian numbers:

$$A(m, l) = (m - l + 1)A(m - 1, l - 1) + lA(m - 1, l)$$

Recursive formula by [Thompson \(1966\)](#):

$$J_n(r) = \frac{n+r}{2(n-1)}J_{n-1}(r-1) + \frac{n-r}{2(n-1)}J_{n-1}(r+1)$$

Connection between them:

$$J_n(r) = \frac{1}{(n-1)!}A\left(n-1, \frac{n+r}{2}\right)$$