

Indecomposable coverings with homothetic polygons

István Kovács

March 12, 2015

Abstract

We prove that for any convex polygon S with at least four sides, or a concave one with no parallel sides, and any $m > 0$, there is an m -fold covering of the plane with homothetic copies of S that cannot be decomposed into two coverings.

1 Introduction

Let $\mathcal{C} = \{ C_i \mid i \in I \}$ be a collection of planar sets. It is an m -fold covering if every point in the plane is contained in at least m members of $\mathcal{C}$. A 1-fold covering is simply called a *covering*.

A planar set S is said to be cover-decomposable if there is a constant $m = m(S)$ such that every m -fold covering of the plane with translates of S can be decomposed into two coverings. J. Pach [3], proposed the problem of determining all cover-decomposable sets in 1980. He conjectured, that all planar convex sets are cover-decomposable. The conjecture has been verified, in several steps, for all convex polygons [9] (see also [4], [11]). However, very recently, Pálvölgyi proved that the unit disc is not cover-decomposable [8]. His result holds also for convex sets with smooth boundary.

The problem of determining cover-decomposable sets has been generalized in many directions, see [5] for a survey.

A *homothetic* transformation is the composition of a translation and a scaling. Keszegh and Pálvölgyi [1] proved that any 12-fold covering of the plane with homothetic copies of a fixed triangle T can be decomposed into two coverings. In this note we prove that, with a few possible exceptions, this result cannot be extended to other polygons.

Theorem 1. *Let S be a convex polygon with at least four sides, or a concave polygon with no parallel sides, and let $m > 0$. There is an m -fold covering of the plane with homothetic copies of S that cannot be decomposed into two coverings.*

For *convex* polygons we can keep the sizes of the homothetic copies “almost equal”.

Theorem 2. *Let S be a convex polygon with at least four sides, let $\varepsilon > 0$ and $m > 0$. There is a collection of homothetic copies of S , each of them with scaling factor between $1 - \varepsilon$ and $1 + \varepsilon$, which forms an m -fold covering of the plane that cannot be decomposed into two coverings.*

Our method is based on the ideas of Pálvölgyi [7], [8].

2 Preparations

Most of the papers about cover-decomposability investigate the problem in its *dual form*.

Suppose that $\mathcal{H} = \{ S_i \mid i \in I \}$ is collection of *translates* of S that form an m -fold covering of the plane. For every $i \in I$, let c_i be the center of gravity of S_i . Let $\mathcal{H}' = \{ c_i \mid i \in I \}$ be the set

This work has been supported by OTKA NN-102029.

of the centers. For any point a , let $-S(a)$ be a translate of $-S$ whose center of gravity is a . Then $a \in S_i$ if and only if $c_i \in -S(a)$. Therefore, the collection $\mathcal{H}$ can be decomposed into two coverings if and only if the points of the set $\mathcal{H}'$ can be colored with two colors, such that every translate of S contains points of both colors. This idea is originally due to J. Pach [4].

If we have homothetic copies, then the dual version of the problem is not equivalent to the original one. However, in this paper we give a tricky definition of the dual form.

Fix a coordinate system and let o be the origin. If it does not lead to confusion, for any point p , we denote its position vector $\vec{op}$ also by p . For any α real, set S , and point p , let

$$\alpha \cdot S(p) = \{\alpha \cdot x + p \mid x \in S\}.$$

The Minkowski sum of any convex polygons S and T is defined as

$$S + T = \{s + t \mid s \in S, t \in T\}.$$

Let S be a fixed convex polygon of at least four sides, $o \in S$. It is well known [10] that for any $\alpha, \beta \geq 0$

$$\alpha \cdot S + \beta \cdot S = (\alpha + \beta) \cdot S.$$

As an easy consequence, we get the following statement.

Statement 1. *Let $\alpha, \beta \geq 0$, $p, q \in \mathbb{R}^2$. $(\alpha + \beta) \cdot S(p)$ contains q if and only if $\alpha \cdot S(p)$ and $-\beta \cdot S(q)$ intersect each other.*

First, for every pair (k, l) , we will construct a collection of homothetic copies of S , $\mathcal{X}_{k,l}$ and a collection of translates of $-S$, $\mathcal{Y}_{k,l}$ with the property that for every red-blue coloring of the elements of $\mathcal{X}_{k,l}$, there is an element of $\mathcal{Y}_{k,l}$ which intersects exactly k elements, all of which are red (resp. exactly l elements, all of which are blue).

Then we “dualize” this construction, for $m = k = l$, as follows. Replace each element of $\mathcal{X}_{m,m}$ by a larger homothetic copy, let $\mathcal{X}'_{m,m}$ be the new collection. Replace each element of $\mathcal{Y}_{m,m}$ by a point, let $\mathcal{Y}'_{m,m}$ be the set of these points.

By Statement 1, $\mathcal{X}'_{m,m}$ and $\mathcal{Y}'_{m,m}$ have the following property.

For every red-blue coloring of the elements of $\mathcal{X}'_{m,m}$, there is an element (point) of $\mathcal{Y}'_{m,m}$ which is contained in exactly m elements of $\mathcal{X}'_{m,m}$, all of which are of the same color.

So, for every m , $\mathcal{X}'_{m,m}$ forms a non-decomposable m -fold covering of the points in $\mathcal{Y}'_{m,m}$. Finally, we extend it to a non-decomposable m -fold covering of the whole plane.

3 Proof of Theorems 1 and 2

Let S be a fixed convex polygon of at least four sides, $o \in S$. We say that o is the *center* of S . We can assume that S is contained in the unit disc of center o . By definition, $-S$ denotes the reflection of S about the origin. Let $v_1, v_2, \dots, v_n$ be the vertices of $-S$, ordered clockwise. Indices are understood mod n , that is, v_{n+1} means v_1 .

Definition 1. For every i , $1 \leq i \leq n$, let E^i denote the convex wedge whose apex is at the origin and its bounding halflines are the translates of $\vec{v_i v_{i-1}}$ and $\vec{v_i v_{i+1}}$. E^i is called the wedge that belongs to vertex v_i of $-S$.

Choose a direction d which is *not parallel* to the sides of S , and the two vertices, v_a and v_b , where S can be touched by a line parallel to d , are not adjacent. Assume without loss of generality that d is horizontal, v_a is the highest, v_b is the lowest vertex of S . Let Q be a quadrilateral created from S by extending the sides at v_a and v_b . Let v_r and v_l be the rightmost and the leftmost vertices of Q , respectively. See Figure 1. We can assume without loss of generality that v_l is not lower than v_r . Indeed, if v_l is lower than v_r , then we can apply a reflection of S about the y -axis. Let $\delta > 0$ be a very small constant.

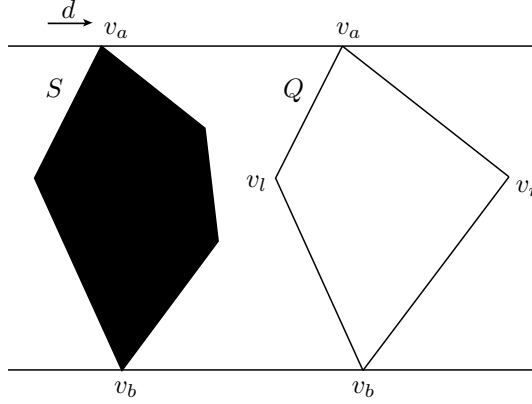


Figure 1: S , Q and the corresponding vertices.

For every pair (k, l) , $k, l \geq 1$, we will construct a triple $\mathcal{T}_{k,l} = (\mathcal{X}_{k,l}, \mathcal{E}_{k,l}^a, \mathcal{E}_{k,l}^b)$, where $\mathcal{X}_{k,l} = \{\varepsilon_i \cdot S(p_i) \mid i \in I_{k,l}\}$, $\varepsilon_i > 0$, a collection of homothetic copies of S , $\mathcal{E}_{k,l}^a = \{E^a(q_j) \mid j \in J_{k,l}^a\}$ and $\mathcal{E}_{k,l}^b = \{E^b(r_j) \mid j \in J_{k,l}^b\}$ are collections of translates of the wedges E^a and E^b , respectively, for some $I_{k,l}, J_{k,l}^a, J_{k,l}^b$ index sets. $\mathcal{T}_{k,l}$ will have the following properties.

Property (1) *For every red-blue coloring of the elements of $\mathcal{X}_{k,l}$, either there is an element of $\mathcal{E}_{k,l}^a$ which intersects exactly k elements of $\mathcal{X}_{k,l}$, all of which are red, or there is an element of $\mathcal{E}_{k,l}^b$ which intersects exactly l elements of $\mathcal{X}_{k,l}$, all of which are blue.*

Property (2) *There is a disc $D_{k,l}$ of radius δ which contains all apices of the wedges in $\mathcal{E}_{k,l}^a$ and $\mathcal{E}_{k,l}^b$, and all elements of $\mathcal{X}_{k,l}$.*

First we define $\mathcal{T}_{k,1}$ and $\mathcal{T}_{1,l}$. For arbitrary k , let $\mathcal{X}_{k,1}$ be k very small homothetic copies of S , very close to each other on a horizontal line. $\mathcal{E}_{k,1}^a$ contains one translate of the wedge E^a that intersects all k homothetic copies, and $\mathcal{E}_{k,1}^b$ contains k translates of the wedge E^b , each intersects exactly one of the k homothetic copies, but each intersects a different one. See Figure 2. We define the triple $\mathcal{T}_{1,l}$ similarly, for any l .

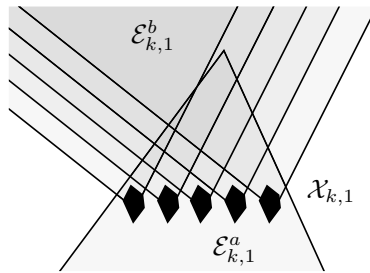


Figure 2: The construction of $\mathcal{T}_{k,1}$.

Suppose now, that we have already defined $\mathcal{T}_{k,l-1}$ and $\mathcal{T}_{k-1,l}$. Take a translate of $\mathcal{T}_{k,l-1}$ so that the center of $D_{k,l-1}$ is $(0, 0)$, and a translate of $\mathcal{T}_{k-1,l}$ so that the center of $D_{k-1,l}$ is $(1, 3\delta)$. Place a suitable homothetic copy $S' = \varepsilon \cdot S$ of S between points $(0, 0)$ and $(1, 3\delta)$ such that

- (i) S' intersects all wedges in $\mathcal{E}_{k,l-1}^b$, and all wedges in $\mathcal{E}_{k-1,l}^a$,
- (ii) S' does not intersect any of the wedges in $\mathcal{E}_{k,l-1}^a$, and any of the wedges in $\mathcal{E}_{k-1,l}^b$.

See Figure 3.

Let

$$\begin{aligned}\mathcal{X}_{k,l} &= \mathcal{X}_{k-1,l} \cup \mathcal{X}_{k,l-1} \cup \{S'\}, \\ \mathcal{E}_{k,l}^a &= \mathcal{E}_{k-1,l}^a \cup \mathcal{E}_{k,l-1}^a, \quad \mathcal{E}_{k,l}^b = \mathcal{E}_{k-1,l}^b \cup \mathcal{E}_{k,l-1}^b.\end{aligned}$$

Apply a suitable scaling, so that Property (2) is satisfied. We claim that Property (1) is also satisfied. Color the elements of $\mathcal{X}_{k,l}$ by red and blue. Suppose that S' is red. In the subconfiguration that corresponds to $\mathcal{T}_{k-1,l}$, either there is a translate of E^a that intersects exactly $k-1$ elements of $\mathcal{X}_{k-1,l}$, all of which are red, or there is a translate of E^b that intersects exactly l elements of $\mathcal{X}_{k-1,l}$, all of which are blue. In the first case, the corresponding translate of E^a intersects exactly one more element of $\mathcal{X}_{k,l}$, S' , and it is red, so we are done. In the second case, the corresponding translate of E^b does not intersect any other element of $\mathcal{X}_{k,l}$, so we are done again. We can argue the same way, if S' is colored blue. Consequently, Property (1) is satisfied.

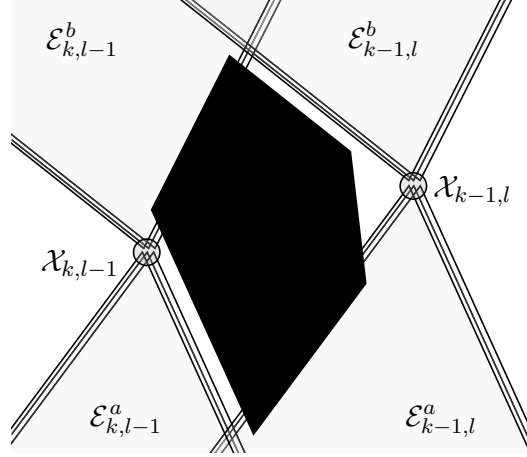


Figure 3: The induction step.

To obtain a non-decomposable m -fold covering, consider $\mathcal{T}_{m,m} = (\mathcal{X}_{m,m}, \mathcal{E}_{m,m}^a, \mathcal{E}_{m,m}^b)$. $\mathcal{X}_{m,m} = \{\varepsilon_i \cdot S(p_i) \mid i \in I_{m,m}\}$, $\varepsilon_i > 0$, a collection of homothetic copies.

Replace each element of $\mathcal{E}_{m,m}^a$ (resp. $\mathcal{E}_{m,m}^b$) by a translate of $-S$ such that its vertex v_a (resp. v_b) moves to its apex. We obtain a collection of translates of $-S$, $\mathcal{Y}_{m,m} = \{-S(q_j) \mid j \in J_{m,m}\}$, with the property that for every red-blue coloring of the elements of $\mathcal{X}_{m,m}$, there is an element of $\mathcal{Y}_{m,m}$ which intersects exactly m elements of $\mathcal{X}_{m,m}$, all of the same color.

Let $\mathcal{X}'_{m,m} = \{(1 + \varepsilon_i) \cdot S(p_i) \mid i \in I_{m,m}\}$, a collection of homothetic copies of S , and let $\mathcal{Y}'_{m,m} = \{q_j \mid j \in J_{m,m}\}$, a collection of points. By Statement 1, for every red-blue coloring of the elements of $\mathcal{X}'_{m,m}$, there is an element (point) of $\mathcal{Y}'_{m,m}$ which is contained in exactly m elements of $\mathcal{X}'_{m,m}$, all of the same color.

That is, $\mathcal{X}'_{m,m}$ forms a non-decomposable m -fold covering of the points in $\mathcal{Y}'_{m,m}$. Moreover, for any $\varepsilon > 0$, if we choose δ small enough, then the scaling factor of each member of $\mathcal{X}'_{m,m}$ is between $1 - \varepsilon$ and $1 + \varepsilon$.

Now we extend $\mathcal{X}'_{m,m}$ to a non-decomposable m -fold covering of the whole plane as follows. We will add homothetic copies of S to $\mathcal{X}'_{m,m}$ that do not contain any point in $\mathcal{Y}'_{m,m}$, but each point in the plane will be covered at least m times. If we allow arbitrary small copies in the covering, then the extension is trivial, since $\mathcal{Y}'_{m,m}$ is a finite point set. Just add *all* homothetic copies of S that do not contain any point of $\mathcal{Y}'_{m,m}$.

If we want to keep the sizes almost equal, we have to be more careful. Points in $\mathcal{Y}'_{m,m}$ are of two types, type *a* (resp. type *b*) is the set of those which come from a wedge in $\mathcal{E}_{m,m}^a$ (resp. $\mathcal{E}_{m,m}^b$). Observe, that any two points of the same type determine a line which is almost horizontal. In fact,

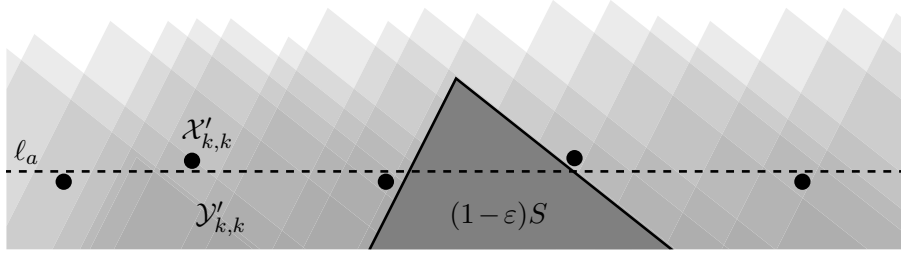


Figure 4: The points of type a in $\mathcal{X}'_{m,m}$ are almost on the line ℓ_a .

we can take two horizontal lines, ℓ_a and ℓ_b , at distance $\text{Vert}(v_a v_b)$, such that all points of type a (resp. type b) are at distance at most δ from ℓ_a (resp. ℓ_b). See Figures 4 and 5.

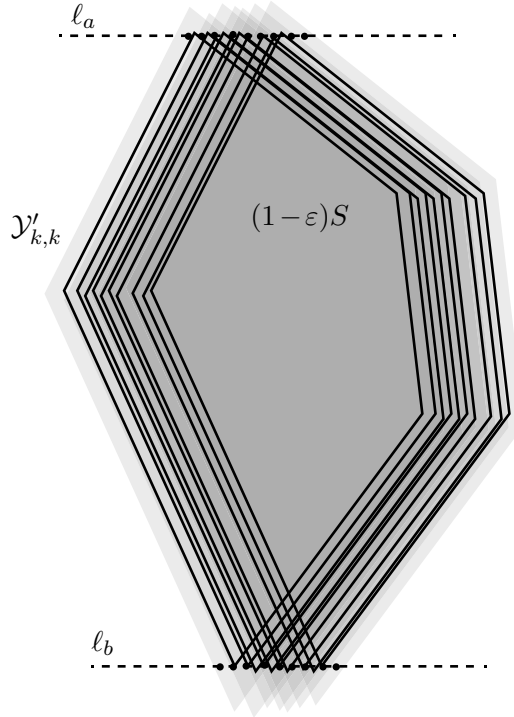


Figure 5: Extending the cover by translates of $(1-\varepsilon)S$.

Add all translates of $(1-\varepsilon)S$ which avoid the points in $\mathcal{Y}'_{m,m}$. Now it is not hard to see that the resulting collection is an m -fold covering of the whole plane, and by the construction of $\mathcal{X}'_{m,m}$ and $\mathcal{Y}'_{m,m}$, it is not decomposable. This concludes the proof of Theorem 2, and also the proof of Theorem 1 in the special case when S is convex.

Now suppose that S is concave with no parallel sides and let $m > 0$. D. Pálvölgyi [7] constructed a collection $\mathcal{X}'_{m,m}$ of *translates* of S , and a set $\mathcal{Y}'_{m,m}$ of points such that $\mathcal{X}'_{m,m}$ forms a non-decomposable m -fold covering of the points in $\mathcal{Y}'_{m,m}$. Add *all* homothetic copies of S that does not contain any point of $\mathcal{Y}'_{m,m}$. The resulting collection is clearly an m -fold covering of the plane, and just like in the previous argument, it is not decomposable. This finishes the proof of Theorem 1.

Remark 1. The dual version of this problem is still open. Let S be a polygon of at least four sides. Is there an $m = m(S)$ with the following property? Any point set $\mathcal{P}$ can be colored with two colors such that if a homothetic copy of S contains at least m points of $\mathcal{P}$, then it contains points of both colors. If S is concave and has no parallel sides, then the answer is NO to this question, even if we use only translates instead of homothetic copies, by the result of Pálvölgyi [7].

On the other hand, if S is convex and we use only translates, then the answer is YES, by [9]. If we do not allow arbitrarily large and arbitrarily small homothetic copies, then the answer is still YES, the proof in [9] works also in this case. But if we allow all homothetic copies, then the problem is unsolved. See [2] for related results.

Remark 2. We can define a hypergraph $\mathcal{H}_{k,l}$ to the pair $(\mathcal{X}_{k,l}, \mathcal{Y}_{k,l})$ in a natural way, elements of $\mathcal{X}_{k,l}$ correspond to the vertices and elements of $\mathcal{Y}_{k,l}$ correspond to the hyperedges, a hyperedge contains a vertex if and only if the corresponding elements intersect each other. The same hypergraph was used by Pálvölgyi in [7] and in [8] to show that some concave polygons and the unit disc are not cover-decomposable.

Remark 3. It was shown in [6], that for every m , there exists an m -fold covering of the plane with axis-parallel rectangles that cannot be decomposed into two coverings. We can slightly strengthen this result.

Theorem 3. *For any $m > 0$, there is an m -fold covering of the plane with axis-parallel rectangles, each with unit horizontal side, that cannot be decomposed into two coverings.*

The proof is almost identical to the proof of Theorem 1. The main difference is that in the induction step, instead of a very small copy of S , we add a very short vertical segment. We omit the details.

Remark 4. We believe that Theorem 2 can be extended to concave polygons with no parallel sides.

Acknowledgment

I would like to thank my supervisor Géza Tóth for all the help and for the many discussions. Without him this article would not have been completed.

References

- [1] B. Keszegh and D. Pálvölgyi. Octants are cover-decomposable. *Discrete & Computational Geometry*, 47(3):598–609, 2012.
- [2] B. Keszegh and D. Pálvölgyi. Convex polygons are self-coverable. *Discrete & Computational Geometry*, 51(4):885–895, 2014.
- [3] J. Pach. Decomposition of multiple packing and covering, 2. In *Kolloquium über Diskrete Geometrie, Salzburg*, pages 169–178, 1980.
- [4] J. Pach. Covering the plane with convex polygons. *Discrete & Computational Geometry*, 1(1):73–81, 1986.
- [5] J. Pach, D. Pálvölgyi, and G. Tóth. Survey on the decomposition of multiple coverings. *Geometry–Intuitive, Discrete and Convex, Bolyai Math. Soc. Studies, I. Bárány et al, eds*, 24:219–259, 2013.
- [6] J. Pach, G. Tardos, and G. Tóth. Indecomposable coverings. In *Discrete Geometry, Combinatorics and Graph Theory*, pages 135–148. Springer, 2007.

- [7] D. Pálvölgyi. Indecomposable coverings with concave polygons. *Discrete & Computational Geometry*, 44(3):577–588, 2010.
- [8] D. Pálvölgyi. Indecomposable coverings with unit discs. *arXiv preprint arXiv:1310.6900*, 2013.
- [9] D. Pálvölgyi and G. Tóth. Convex polygons are cover-decomposable. *Discrete & Computational Geometry*, 43(3):483–496, 2010.
- [10] R. Schneider. *Convex bodies: the Brunn-Minkowski theory: Chapter 3 Minkowski addition*, volume 44. Cambridge University Press, 1993.
- [11] G. Tardos and G. Tóth. Multiple coverings of the plane with triangles. *Discrete & Computational Geometry*, 38(2):443–450, 2007.