



Generalized heat conduction in heat pulse experiments



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ABSTRACT

A novel equation of heat conduction is derived with the help of a generalized entropy current and internal variables. The obtained system of constitutive relations is compatible with the momentum series expansion of the kinetic theory. The well known Fourier, Maxwell–Cattaneo–Vernotte, Guyer–Krumhansl, Jeffreys-type, and Cahn–Hilliard type equations are derived as special cases.

Some remarkable properties of solutions of the general equation are demonstrated with heat pulse initial and boundary conditions. A simple numerical method is developed and its stability is proved. Apparent faster than Fourier pulse propagation is calculated in the over-diffusion regime.

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1. Introduction

Recently a common generalization of the Fourier, Maxwell–Cattaneo–Vernotte, Guyer–Krumhansl, Jeffreys-type and Green–Naghdi heat conduction equations was derived in the framework of non-equilibrium thermodynamics [1]. Then experimental and theoretical studies were performed in order to understand the role of different terms and also the possibility of detecting non-Fourier effects [2–4]. According to the basic hypothesis of these investigations, material heterogeneities are manifested in additional higher order space and time derivatives in the material functions and result in nonlocal and memory effects (see e.g. [5–7]). However, these effects may be not apparent, the observed phenomena may be Fourier like due to the universal dissipative nature of the additional terms. Therefore it is important to identify and analyze possible qualitative signatures for experimental observation.

The non-equilibrium thermodynamical theory of generalized heat conduction of [1] is based on the assumption of a minimal deviation from local equilibrium. The deviation is expressed in terms of new fields and may appear both in the density and in the current density of the entropy:

- In the entropy a quadratic expression of a vectorial internal variable represents the deviation from local equilibrium in the continua [8,9]. This contribution results in memory effects.

- A generalization of the entropy current density, with the help of current multipliers, represents the deviation of the currents from their local equilibrium form [10–13]. This contribution results in nonlocal effects.

The modifications are restricted only by the second law of thermodynamics, do not incorporate assumptions about the structure of the continua, therefore in this sense the approach is universal [14,15].

An important problematic point of the theory is, that the structure of the derived system of evolution equations seems to be incompatible with the existing theories of Extended Thermodynamics, that is with the hyperbolic system of the momentum series expansion hierarchy of the kinetic theory [16]. In particular it does not compatible with the ballistic phonons, a well explained non-Fourier propagation mechanism in low temperature materials [17,18].

In this paper we slightly modify and extend the approach of [1] introducing the heat flux as a basic field, instead of the general vectorial internal variable of [1]. We also introduce an additional second order tensorial internal variable and the corresponding generalization of the entropy flux by current multipliers. This way we reproduce the first two levels of the hierarchy of kinetic theory in a generalized, phenomenological framework, without any particular assumptions on the structure of the material (e.g. a rarefied gas). We assume only a second law compatible deviation from local equilibrium.

What we obtain is more general than the corresponding set of equations of Extended Thermodynamics, that is the equations obtained from or motivated by the hierarchy of moments in kinetic

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theory. Due to the phenomenological assumptions the whole structure is flexible and we can derive several known generalizations of the Fourier equation in a uniform framework obtaining information regarding their applicability and interrelations. In this respect it is remarkable that Green–Naghdi equations [19–21] are obtained as well as Cahn–Hilliard type heat conduction [22,23]. These heat conduction models were justified by rigorous mathematical methods but not related to Extended Thermodynamics.

Another important property of our approach is that old paradoxes and reservations regarding some forms of heat conduction are shown in a new light. For example the well discussed paradox of heat waves with negative values of temperature of the Maxwell–Cattaneo–Vernotte and the Jeffreys-type equations (see e.g. in [24,25]) seems to be removed simply because thermodynamics requires the gradient of the reciprocal temperature instead of the gradient of the temperature in the related terms of the equations.

This paper focuses on the problem of observability of non-Fourier heat conduction from a theoretical point of view. Solving generalized heat conduction models with heat pulse initial and boundary conditions will demonstrate that Fourier type solutions may appear unexpectedly and therefore in addition of wavelike effects one may look for other observable benchmarks of heat conduction beyond Fourier.

In the next section we introduce the theory and derive the heat conduction equation up to the second current multiplier and show some known particular cases. The third section describes the heat pulse experiment, then we introduce a simple finite difference numerical method to solve the set of equations. Finally, in the fifth section, we show some demonstrative solutions of the equations on the example of laser flash experiment in order to identify possible non-Fourier effects.

2. Non-equilibrium thermodynamics of heat conduction

In this paper we restrict ourselves to rigid heat conductors, therefore the time derivatives are partial and the density of the material is constant. Our starting point is the balance of internal energy:

$$\partial_t e + \nabla \cdot \mathbf{q} = 0. \quad (1)$$

Here e is the density of the internal energy, and $\mathbf{q}$ is the heat flux, the current density of the internal energy. ∂_t denotes the partial time derivative and ∇ with the central dot is the divergence, $\nabla \cdot \mathbf{q} = \text{tr}(\nabla \mathbf{q})$.

The second law is given in the following form

$$\partial_t s + \nabla \cdot \mathbf{J} \geq 0. \quad (2)$$

Here s is the entropy density and $\mathbf{J}$ is the entropy current density vector. For modeling phenomena beyond local equilibrium, we introduce the heat flux $\mathbf{q}$ as basic field variable and also a second order tensorial internal variable denoted by $\mathbf{Q}$. The advantage of using the heat flux as basic field quantity instead of a vectorial internal variable of the treatment in [1] is the easier comparison with Extended Thermodynamics. The deviation from local equilibrium will be characterized by two basic constitutive hypotheses:

- We assume a quadratic dependence of the entropy density on the additional fields [8]:

$$s(e, \mathbf{q}, \mathbf{Q}) = s_{eq}(e) - \frac{m_1}{2} \mathbf{q} \cdot \mathbf{q} - \frac{m_2}{2} \mathbf{Q} : \mathbf{Q}, \quad (3)$$

where m_1 and m_2 are positive constant material coefficients. This is not a complete isotropic representation, for the sake of simplicity we have introduced a single material coefficient for the second order tensor $\mathbf{Q}$, too. The derivative of the local equilibrium part of the entropy function s_{eq} by the internal

energy is the reciprocal temperature: $\frac{ds_{eq}}{de} = \frac{1}{T}$ and $\mathbf{Q} : \mathbf{Q} = \text{tr}(\mathbf{Q} \cdot \mathbf{Q})$. The quadratic form may be considered as a first approximation in case of the heat flux and is due to the Morse lemma for the internal variable [9]. The sign is determined requiring concave entropy function, that is, thermodynamic stability [18,17].

- We assume that the entropy flux is zero if $\mathbf{q} = 0$ and $\mathbf{Q} = 0$. Therefore it can be written in the following form:

$$\mathbf{J} = \mathbf{b} \cdot \mathbf{q} + \mathbf{B} : \mathbf{Q}. \quad (4)$$

Here $\mathbf{b}$ is a second order tensorial constitutive function and $\mathbf{B}$ is a third order one. They are the current multipliers introduced by Nyíri [11]. General aspects of this assumption were treated in [12] and the special case of heat conduction was considered in [1,6].

Now the basic fields are $T, \mathbf{q}$ and $\mathbf{Q}$, the constitutive functions are $\mathbf{b}$ and $\mathbf{B}$. The entropy production is:

$$\begin{aligned} \partial_t s + \nabla \cdot \mathbf{J} = & -\frac{1}{T} \nabla \cdot \mathbf{q} - m_1 \mathbf{q} \cdot \partial_t \mathbf{q} - m_2 \mathbf{Q} : \partial_t \mathbf{Q} + \mathbf{b} : \nabla \mathbf{q} + \mathbf{q} \cdot (\nabla \cdot \mathbf{b}) \\ & + \mathbf{B} : \nabla \mathbf{Q} + \mathbf{Q} : (\nabla \cdot \mathbf{B}) = \left(\mathbf{b} - \frac{1}{T} \mathbf{I} \right) : \nabla \mathbf{q} + (\nabla \cdot \mathbf{b} - m_1 \partial_t \mathbf{q}) \cdot \mathbf{q} \\ & + (\nabla \cdot \mathbf{B} - m_2 \partial_t \mathbf{Q}) : \mathbf{Q} + \mathbf{B} : \nabla \mathbf{Q} \geq 0. \end{aligned} \quad (5)$$

Here $\mathbf{I}$ is the unit tensor and the triple dot denotes the full contraction of third order tensors. In the last two rows the first and the third terms are products of second order tensors, the second term is the product of vectors and the last term is of third order ones. The time derivatives of the state variables $\mathbf{q}$ and $\mathbf{Q}$ represent their evolution equations, here they are constitutive quantities. Therefore one can identify four thermodynamic forces and currents in the above expression and assume linear relationship between them in order to obtain the solution of the entropy inequality.

The third and fourth force–current pairs are related to the tensorial internal variable $\mathbf{Q}$. In case of isotropic materials only the second order tensors can show cross effects (extended thermal and internal interactions), the vectorial (thermal) and third order tensorial terms (extended internal) are independent (see Table 1).

In the following we will simplify the treatment and develop the theory in one spatial direction. In the one dimensional representation of the tensors we remove the boldface letters, and the one dimensional spatial derivative is denoted by ∂_x .

In this case the entropy production can be rewritten as:

$$\left(b - \frac{1}{T} \right) : \partial_x q + (\partial_x b - m_1 \partial_t q) q + (\partial_x B - m_2 \partial_t Q) Q + B \partial_x Q \geq 0. \quad (6)$$

The linear relations between the thermodynamic fluxes and forces result in the following constitutive equations:

$$m_1 \partial_t q - \partial_x b = -l_1 q, \quad (7)$$

$$m_2 \partial_t Q - \partial_x B = -k_1 Q + k_{12} \partial_x q, \quad (8)$$

$$b - \frac{1}{T} = -k_{21} Q + k_2 \partial_x q, \quad (9)$$

$$B = n \partial_x Q. \quad (10)$$

The entropy inequality (6) requires the following inequalities

$$l_1 \geq 0, \quad k_1 \geq 0, \quad k_2 \geq 0, \quad n \geq 0, \quad \text{and} \quad K = k_1 k_2 - k_{12} k_{21} \geq 0. \quad (11)$$

The above set of constitutive Eqs. (7)–(10) together with the energy balance (1) and the caloric equation of state $T(e)$ give a solvable set of equations, with suitable boundary and initial conditions. In case of constant coefficients one can easily eliminate the current multipliers by substituting them from Eqs. (9) and (10) into Eqs. (7) and (8) and obtain:

Table 1
Thermodynamic fluxes and forces.

	Thermal	Extended thermal	Internal	Extended internal
Fluxes	$\nabla \cdot \mathbf{b} - m_1 \partial_t \mathbf{q}$	$\mathbf{b} - \frac{1}{T} \mathbf{I}$	$\nabla \cdot \mathbf{B} - m_2 \partial_t \mathbf{Q}$	$\mathbf{B}$
Forces	$\mathbf{q}$	$\nabla \mathbf{q}$	$\mathbf{Q}$	$\nabla \mathbf{Q}$

$$m_1 \partial_t q + l_1 q - k_2 \partial_x^2 q = \partial_x \frac{1}{T} - k_{21} \partial_x Q, \quad (12)$$

$$m_2 \partial_t Q + k_1 Q - n \partial_x^2 Q = k_{12} \partial_x q. \quad (13)$$

Here ∂_x^i denotes the i th partial derivative by x . This set of equations is similar to the 13 field equations of kinetic theory [26].

We can eliminate the internal variable Q too, and obtain the following general constitutive equation for the heat flux q :

$$m_1 m_2 \partial_{tt} q + (m_2 l_1 + m_1 k_1) \partial_t q - (m_1 n + m_2 k_2) \partial_{xx} q + n k_2 \partial_x^4 q - (l_1 n + K) \partial_{xx} q + k_1 l_1 q = m_2 \partial_{xt} \frac{1}{T} + k_1 \partial_x \frac{1}{T} - n \partial_x^3 \frac{1}{T}. \quad (14)$$

In this equation the coefficients are nonnegative according to the second law. One can distinguish between the following special cases:

2.1

$n = 0$. Then the second current multiplier B is eliminated and therefore the highest order spatial derivatives of q and T are missing.

$$m_1 m_2 \partial_{tt} q + (m_2 l_1 + m_1 k_1) \partial_t q - m_2 k_2 \partial_{xx} q + k_1 l_1 q - K \partial_{xx} q = m_2 \partial_{xt} \frac{1}{T} + k_1 \partial_x \frac{1}{T}. \quad (15)$$

2.2. Ballistic-conductive

$n = k_2 = 0$. Then we obtain

$$m_1 m_2 \partial_{tt} q + (m_2 l_1 + m_1 k_1) \partial_t q + k_1 l_1 q - k_{12} k_{21} \partial_{xx} q = m_2 \partial_{xt} \frac{1}{T} + k_1 \partial_x \frac{1}{T}. \quad (16)$$

We will see, that this equation may show ballistic propagation. In this case either k_{12} or k_{21} is not positive because of (11).

2.3. Guyer–Krumhansl

$n = m_2 = 0$. In this case both the tensorial internal variable and the corresponding current multiplier is eliminated:

$$m_1 k_1 \partial_t q + k_1 l_1 q - K \partial_{xx} q = k_1 \partial_x \frac{1}{T}. \quad (17)$$

2.4. Generalized

$n = m_1 = 0$. Then one obtains the generalized heat conduction law of [1]:

$$m_2 l_1 \partial_t q - m_2 k_2 \partial_{xx} q - K \partial_{xx} q + k_1 l_1 q = m_2 \partial_{xt} \frac{1}{T} + k_1 \partial_x \frac{1}{T}. \quad (18)$$

2.5. Cahn–Hilliard type

$n = m_1 = m_2 = 0$. Then one obtains Fourier equation extended by the second spatial derivative of the heat flux:

$$k_1 l_1 q - K \partial_{xx} q = k_1 \partial_x \frac{1}{T}. \quad (19)$$

This equation is a Cahn–Hilliard type one, similar to the hypertextemperature model of Forest and Amestoy except a sign [23].

2.6. Jeffreys type

$n = m_1 = k_2 = 0$ and k_{12} or $k_{21} = 0$. Then one obtains the Jeffreys type like heat conduction law:

$$m_2 l_1 \partial_t q + k_1 l_1 q = m_2 \partial_{xt} \frac{1}{T} + k_1 \partial_x \frac{1}{T}. \quad (20)$$

Strictly speaking it is different of the Jeffreys type equation, because of the reciprocal temperature instead of the temperature on the right hand side.

2.7. MCV

$n = m_2 = k_2 = 0$ and k_{12} or $k_{21} = 0$. Then we get the Maxwell–Cattaneo–Vernotte equation:

$$m_1 \partial_t q + l_1 q = k_1 \partial_x \frac{1}{T}. \quad (21)$$

2.8. Fourier

$n = m_2 = m_1 = k_2 = 0$ and k_{12} or $k_{21} = 0$. Then eliminating almost everything one obtains the thermodynamic form of the Fourier law:

$$l_1 q = \partial_x \frac{1}{T} = -\frac{1}{l_1 T^2} \partial_x T. \quad (22)$$

Here $\lambda = \frac{1}{l_1 T^2}$ is the Fourier heat conduction coefficient.

3. Heat pulse experiment

Heat pulse experiments were important in the discovery of second sound and ballistic propagation phenomena (see e.g. [27,28]) and in the framework of laser flash method it is an important part of standard engineering practice [29–31]. In this section we analyze some consequences of the previous general equations in case of heat pulses in order to identify possible qualitative effects. The system of equations to be solved are the energy balance (1) and the constitutive evolution equations for the heat flux and the internal variable Eqs. (12) and (13) with the initial and boundary conditions specific for the heat pulse experiment. We substitute the caloric equation of state $e = \rho c T$, into the energy balance, where ρ is the density and c is the specific heat. In this numerical study we restrict ourselves to the ballistic-conductive model, where $n = 0$ and $k_2 = 0$. Introducing a convenient notation for the constitutive equations results in:

$$\rho c \partial_t T + \partial_x q = 0, \quad (23)$$

$$\tau_q \partial_t q + q = -\lambda \partial_x T - \kappa_{21} \partial_x Q, \quad (24)$$

$$\tau_Q \partial_t Q + Q = \kappa_{12} \partial_x q. \quad (25)$$

Here $\tau_q = \frac{m_1}{l_1}$ and $\tau_Q = \frac{m_2}{k_1}$ are relaxation times, $\lambda = \frac{1}{T_0^2 l_1}$ is the Fourier heat conduction coefficient and $\kappa_{21} = \frac{k_{21}}{l_1}$, $\kappa_{12} = \frac{k_{12}}{k_1}$ are material parameters.

The front side boundary condition is a heat pulse of the following form:

$$q_0(t) = q(x=0, t) = \begin{cases} q_{\max} \left(1 - \cos\left(2\pi \cdot \frac{t}{t_p}\right)\right) & \text{if } 0 < t \leq t_p, \\ 0 & \text{if } t > t_p. \end{cases}$$

At the backside of the sample, $x = L$, we consider adiabatic insulation, therefore $q(x = L, t) = 0$. Initially the fields are homogeneous and the initial conditions are $T(x, t = 0) = T_0$ and $q(x, t = 0) = 0$.

After these operations we introduce the dimensionless variables $\hat{t}, \hat{x}, \hat{T}, \hat{q}, \hat{Q}$ for the time, space, temperature, heat flux and internal variable, respectively.

$$\begin{aligned} \hat{t} &= \frac{\alpha t}{L^2}, \quad \text{where } \alpha = \frac{\lambda}{\rho c}; \quad \hat{x} = \frac{x}{L}; \\ \hat{T} &= \frac{T - T_0}{T_{\text{end}} - T_0}, \quad \text{where } T_{\text{end}} = T_0 + \frac{\bar{q}_0 t_p}{\rho c L}; \\ \hat{q} &= \frac{q}{\bar{q}_0}, \quad \text{where } \bar{q}_0 = \frac{1}{t_p} \int_0^{t_p} q_0(t) dt; \quad \hat{Q} = \sqrt{-\frac{\kappa_{12}}{\kappa_{21}}} \bar{q}_0 Q. \end{aligned} \quad (26)$$

The dimensionless parameters are

$$\hat{\tau}_\Delta = \frac{\alpha t_p}{L^2}; \quad \hat{\tau}_q = \frac{\alpha \tau_q}{L^2}; \quad \hat{\tau}_Q = \frac{\alpha \tau_Q}{L^2}; \quad \hat{\kappa} = \frac{\sqrt{-\kappa_{12} \kappa_{21}}}{L}. \quad (27)$$

Here $\hat{\kappa} > 0$ and the coefficient in $\hat{Q}$ is positive due to the entropy inequality. Finally we get the ballistic-conductive equations in a non-dimensional form

$$\begin{aligned} \hat{\tau}_\Delta \partial_{\hat{t}} \hat{T} + \partial_{\hat{x}} \hat{q} &= 0, \\ \hat{\tau}_q \partial_{\hat{t}} \hat{q} + \hat{q} + \hat{\tau}_\Delta \partial_{\hat{x}} \hat{T} + \hat{\kappa} \partial_{\hat{x}} \hat{Q} &= 0, \\ \hat{\tau}_Q \partial_{\hat{t}} \hat{Q} + \hat{Q} + \hat{\kappa} \partial_{\hat{x}} \hat{q} &= 0. \end{aligned} \quad (28)$$

If $\hat{\tau}_Q = 0$, then the Guyer–Krumhansl system is obtained.

The corresponding initial conditions are $\hat{T}(\hat{x}, \hat{t} = 0) = 0$, $\hat{q}(\hat{x}, \hat{t} = 0) = 0$ and $\hat{Q}(\hat{x}, \hat{t} = 0) = 0$. The boundary conditions are given for the heat flux only. At the rear side $\hat{q}(\hat{x} = 1, \hat{t}) = 0$ and the heat pulse at the front side is

$$\hat{q}(\hat{x} = 0, \hat{t}) = \begin{cases} \left(1 - \cos\left(2\pi \frac{\hat{t}}{\hat{\tau}_\Delta}\right)\right) & \text{if } 0 < \hat{t} \leq \hat{\tau}_\Delta, \\ 0 & \text{if } \hat{t} > \hat{\tau}_\Delta. \end{cases}$$

The above set of initial and boundary conditions will be sufficient for the numerical solution of the problem.

4. Numerical method

In this and also in the following sections the previously introduced dimensionless variables and parameters will be used without hat. Now we analyze a finite difference method of solution of the above mathematical problem.

We apply a finite difference scheme which is forward in time and one-sided in space. In fact, it uses the values like a centered scheme in space which holds only for shifted fields. There are two types of discretized fields, one of them is covering the space interval from $(0, 1)$, while the others are shifted by $\frac{\Delta x}{2}$ (see Fig. 1).

The essential aspect of this method is that we can neglect the boundary conditions of shifted fields, because the points of these fields correspond to the middle point of each cell. In the particular case of the laser flash experiment it is straightforward to use only the heat flux boundary condition. It is noted that the fields can be interchanged, the shifting is not necessary and it can be mixed, e.g. if one want to define conditions both for temperature and heat flux, then only the current density of heat flux should be shifted.

The discrete form of the balance of internal energy reads as:

$$T_j^{n+1} = T_j^n - \frac{\Delta t}{\tau_\Delta \Delta x} (q_{j+1}^n - q_j^n).$$

The discrete constitutive equations are:

$$q_j^{n+1} = q_j^n \left(1 - \frac{\Delta t}{\tau_q}\right) - \frac{\Delta t \tau_\Delta}{\tau_q \Delta x} (T_j^n - T_{j-1}^n) - \frac{\kappa \Delta t}{\tau_q \Delta x} (Q_j^n - Q_{j-1}^n),$$

$$Q_j^{n+1} = Q_j^n - \frac{\Delta t}{\tau_Q} Q_j^n + \frac{\kappa \Delta t}{\tau_Q \Delta x} (q_{j+1}^n - q_j^n).$$

Here n and j are the time and the space indices.

The scheme is explicit therefore the stability is an essential side of the analysis. We have applied the von Neumann method, i.e. we assumed the solution of the difference equation in the form $T_j^n = \psi^n \cdot e^{ikj\Delta x}$, where ψ is the growth factor with stability condition $|\psi| \leq 1$, i is the imaginary unit, k is the wave number. After substitution we get the system of linear algebraic equations:

$$\mathbf{M} \cdot \begin{pmatrix} T \\ q \\ Q \end{pmatrix} = \mathbf{0},$$

where

$$\mathbf{M} = \begin{pmatrix} 1 - \psi & -\frac{\Delta t}{\tau_\Delta \Delta x} (e^{ik\Delta x} - 1) & 0 \\ -\frac{\Delta t \tau_\Delta}{\tau_q \Delta x} (1 - e^{-ik\Delta x}) & 1 - \frac{\Delta t}{\tau_q} - \psi & -\frac{\kappa \Delta t}{\tau_q \Delta x} (1 - e^{-ik\Delta x}) \\ 0 & \frac{\kappa \Delta t}{\tau_Q \Delta x} (e^{ik\Delta x} - 1) & 1 - \frac{\Delta t}{\tau_Q} - \psi \end{pmatrix}.$$

The characteristic equation of the system can be calculated from $\det \mathbf{M} = 0$ and Jury criteria is applied for stability analysis.

4.1. Jury criteria

If the characteristic equation is given in the form

$$F(\psi) = c_3 \psi^3 + c_2 \psi^2 + c_1 \psi + c_0, \quad (29)$$

where c_i , ($i = 0, 1, 2, 3$), then the related conditions of stability are [32]

1. $F(\psi = 1) > 0$,
2. $F(\psi = -1) < 0$,
3. $|c_0| < c_3$,
4. $|b_0| > |b_2|$, where $b_0 = \begin{vmatrix} c_0 & c_3 \\ c_3 & c_0 \end{vmatrix}$ and $b_2 = \begin{vmatrix} c_0 & c_1 \\ c_3 & c_2 \end{vmatrix}$.

The analytical calculation of these conditions is straightforward, but inconvenient even in this size, so the numerical evaluation of the criteria is recommended. The conditions are checked during the calculations. We did not encounter remarkable stability problems with this discretization.

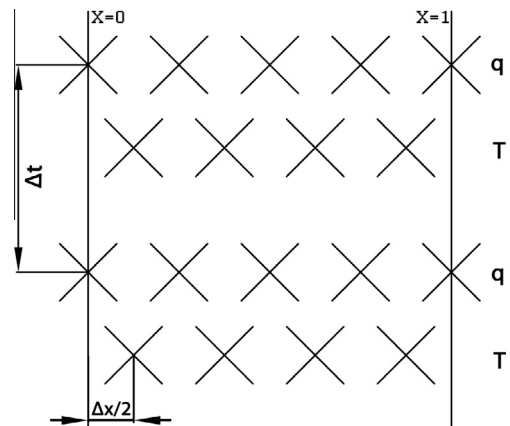


Fig. 1. Discretization method. The crosses denote the points of space discretization.

5. Solutions

In this section we classify the solutions of the Guyer–Krumhansl equation as Fourier-like, MCV-like (under-diffusive) and GK-like (over-diffusive). We also present solutions of the ballistic-conductive system. The characteristics of the solution are demonstrated separately on the backside temperature profile.

5.1. Solutions of the GK-equation

Guyer–Krumhansl equation is a special form of (28) if there are no inertial effects in the propagation of the internal variable Q , that is $\tau_Q = 0$.

Fourier solutions are obtained, if $\tau_q = \kappa^2$. Fig. 2 shows the case when $\tau_q = \kappa^2 = 0.02$ and $\tau_\Delta = 0.04$.

If the dissipation term is small, one can identify the properties of the MCV-equation, see Fig. 3. In this case we solved the equation for $\tau_\Delta = 0.04$, $\tau_q = 0.02$ and $\kappa^2 = 10^{-4}$. $\kappa^2 = 0$ results in exactly the MCV solution.

In the third case ($\tau_\Delta = 0.04$, $\tau_q = 0.02$, $\kappa^2 = 0.04$), the solution is over-diffusive. Remarkable is the speed of the signal propagation (Fig. 4). One can see that the temperature signal arrives earlier than the measurable signal of the Fourier case and the change of the backside temperature is steeper, compared to the Fourier solution.

5.2. Solutions of the ballistic-conductive system

Analysis of the entropy production Eqs. (5) and (14) suggests us ideas to distinguish solution classes with respect of parameters. From Eq. (28) one can see that the parameter $\kappa = 0$ leads to MCV solutions, because it decouples the last two constitutive equations. If it does not hold, there is an additional dissipative term in the entropy production. Let us see these cases more closely.

- **Fourier solution:** In this case we need relatively small τ_q and κ but large τ_Q . It results in that $\kappa \approx 0$, $\tau_Q + \tau_q \approx \tau_Q$. However, we have to note it is only an approximation of Fourier solution, because the term of double time derivative of heat flux in Eq. (14) never can be zero. The particular parameters on Fig. 5 are: $\tau_q = 0.002$; $\tau_Q = 1$; $\tau_\Delta = 0.04$; $\kappa = 0.001$.
- **MCV-kind solution:** This case is very simple, we need only $\kappa = 0$ which results in that the signal propagation will be independent of τ_Q and $K = 0$. Thus, the parameters are: $\tau_\Delta = 0.04$, $\tau_q = 0.02$, $\kappa = 0$. The solution is shown on Fig. 6.

- **GK solution:** The characteristics are same as before; to reach this we need to apply high κ and relaxation times are around the same order of magnitude, so we used $\tau_Q = 0.001$, $\tau_\Delta = 0.04$, $\tau_q = 0.02$, $\kappa = 0.25$; see Fig. 7.
- **Ballistic solutions:** In this case we can observe two propagation speeds, but their detection in simulations is not so easy because of dissipation. For example the particular parameters $\tau_q = 1.9$, $\tau_Q = 0.07$, $\kappa = 0.28$, $\tau_\Delta = 0.065$ results in such a propagation which can be seen on Fig. 8.

6. Discussion

There are some remarkable properties of the thermodynamic approach and the obtained heat conduction model.

- The method of generalization with the help of current multipliers is a second order weakly nonlocal extension of the thermodynamic framework and it is compatible with the more rigorous exploitation methods of the second law [13,6].
- The derived MCV and Jeffreys type constitutive equations, (21) and (20), are different of their usual forms because of the apparent nonlinearity introduced by the gradient of the reciprocal temperature instead of the gradient of the temperature. A possible consequence of this variation is that the unphysical negative temperature solutions of the original MCV and Jeffreys type equations (see e.g. [25]) may be missing for (21) and (20).
- There are several ways to recover Fourier like solutions. Choosing most of the material coefficients zero is not the only way. The general heat conduction equation results in Fourier solutions in many different coefficient combinations, due to the hierarchical structure of the theory. We have demonstrated this on the example of the Guyer–Krumhansl equation. We have distinguished an under-diffusive or wave like regime, when $\tau_q > \kappa^2$ and an over-diffusive regime, when $\tau_q < \kappa^2$. These kind of solutions are observed also in case of the Jeffreys type equation [33], where it is considered as unphysical [34]. Let us remark that neither the original dual-phase lag model nor related differential constitutive equations are satisfactory from a stability point of view [36] and also show other kind of inconsistencies [35]. Our thermodynamic framework is not compatible with the dual-phase lag approach and as a special case results in the constitutive equation (20), that is similar to the Jeffreys type one.

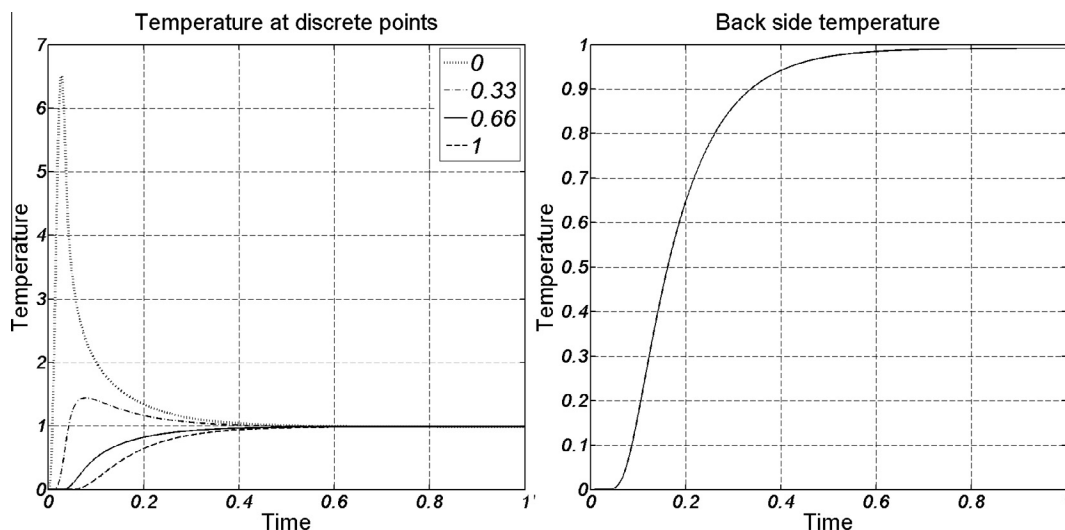


Fig. 2. Fourier-like solution of the GK-equation, $\tau_\Delta = 0.04$, $\tau_q = \kappa^2 = 0.02$.

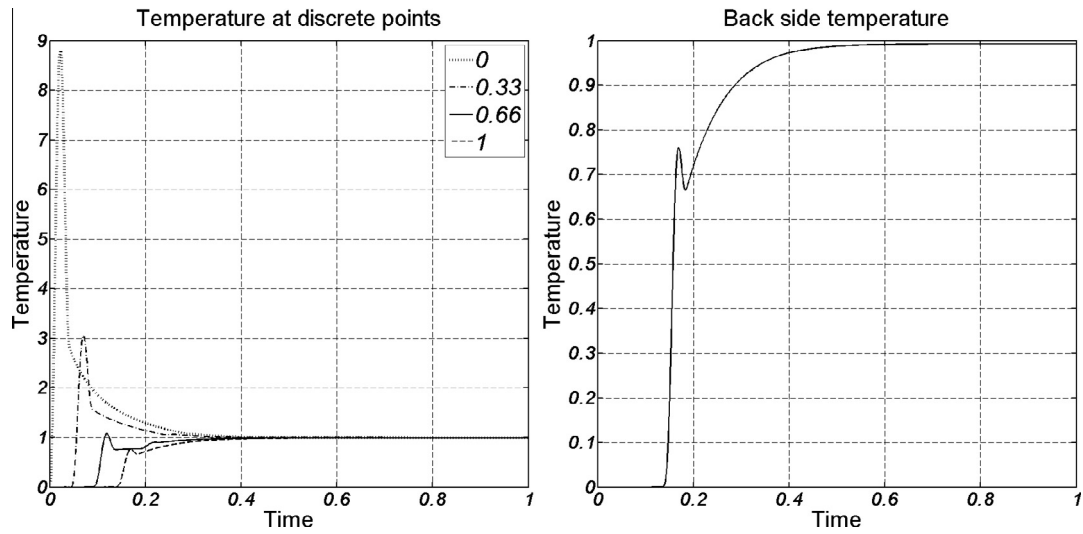


Fig. 3. MCV-like solution of the GK-equation; $\tau_\Delta = 0.04$, $\tau_q = 0.02$, $\kappa^2 = 10^{-4}$.

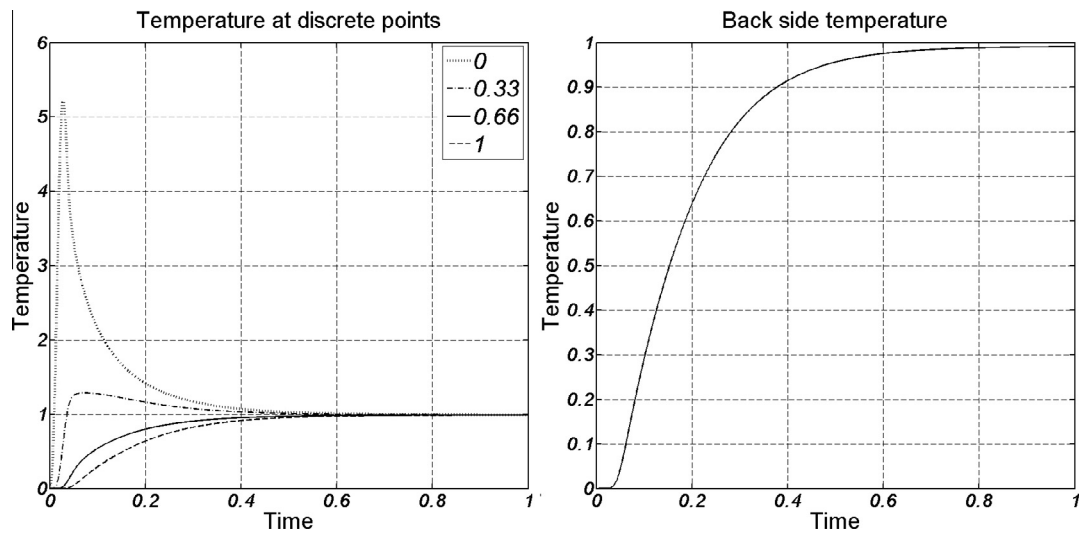


Fig. 4. Over-diffusive solution of the GK-equation; $\tau_\Delta = 0.04$, $\tau_q = 0.02$, $\kappa^2 = 0.04$.

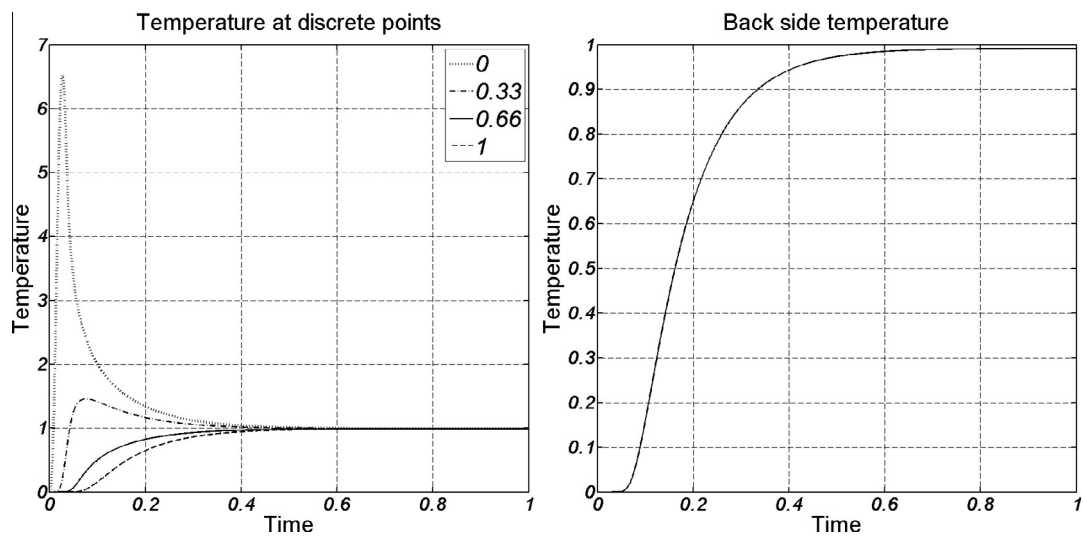


Fig. 5. Fourier-like solution of the ballistic-conductive system; $\tau_Q = 1$, $\tau_\Delta = 0.04$, $\tau_q = 0.002$, $\kappa = 0.001$.

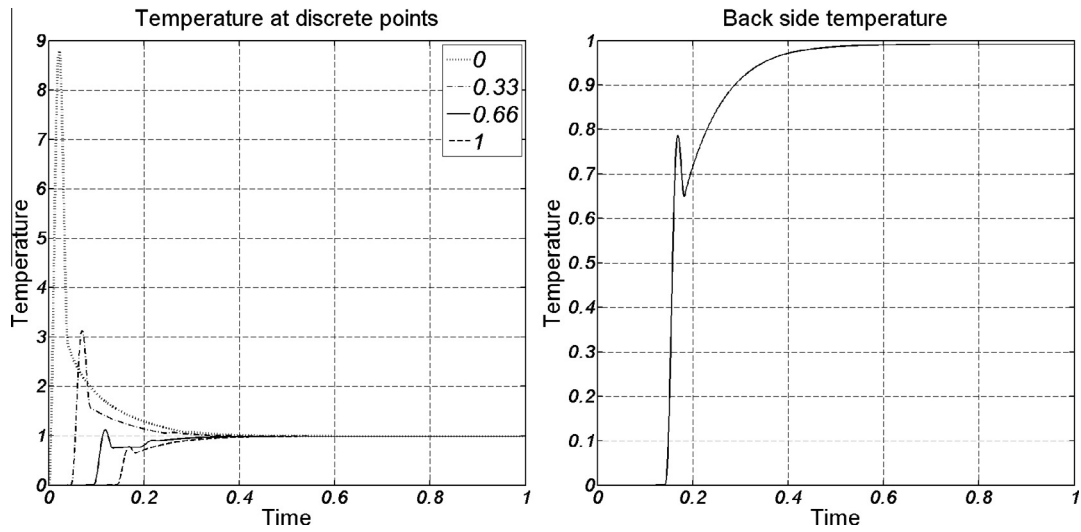


Fig. 6. MCV-like solution of the ballistic-diffusive system, $\tau_A = 0.04$, $\tau_q = 0.02$, $\kappa = 0$.

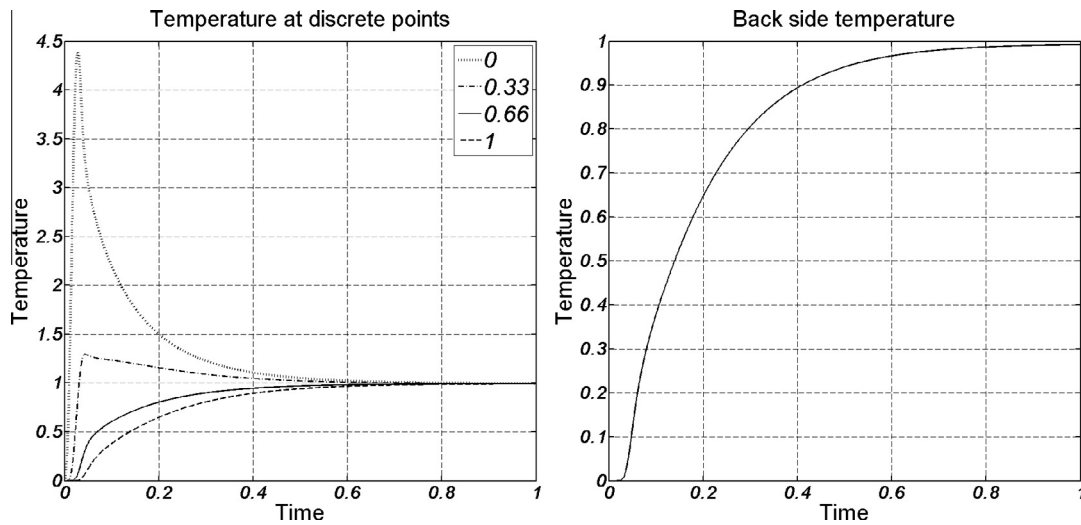


Fig. 7. Guyer-Krumhansl solution of the ballistic-diffusive system; $\tau_Q = 0.001$, $\tau_A = 0.04$, $\tau_q = 0.02$, $\kappa = 0.25$.

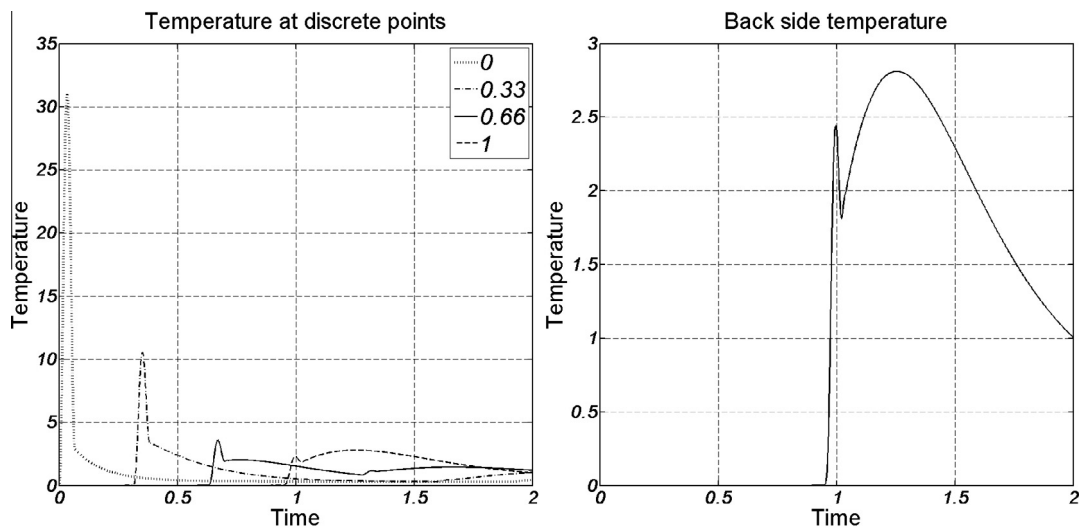


Fig. 8. Two propagation speeds solving the ballistic-diffusive system; $\tau_q = 1.9$, $\tau_Q = 0.07$, $\kappa = 0.28$, $\tau_A = 0.065$.

- The apparent faster than Fourier signal propagation of the over-damped case (see Fig. 4) is a remarkable qualitative effect, contrary to the expected phenomena of the MCV equation where the apparent propagation speed is slower.
- Kinetic theory results in correct ballistic propagation using at least 30 moments, because the material parameters are fixed by the microstructure and the introduced particular microscopical processes [26]. In our phenomenological approach it seems to be no problem to obtain the required ballistic wave speeds in the framework of our three field theory, $(e, \mathbf{q}, \mathbf{Q})$. (13 field in Extended Thermodynamics).

Conflict of interest

None declared.

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